

6.4 Loci, regions and $z \rightarrow w$ transformations

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS locus • perpendicular bisector • Möbius transformation

Loci, regions and $z \rightarrow w$ transformations

Objective

Describe and sketch standard loci in the Argand diagram (circles, bisectors, Apollonius circles, argument half-lines) with their interior regions, and find the image of a locus under a Möbius transformation $w = \frac{az + b}{cz + d}$ by making z the subject and equating moduli or arguments (FP2 3.3–3.4).

1 Key knowledge and worked examples

Locus toolkit. $|z - a| = b$: circle, centre a , radius b . $|z - a| = |z - b|$: perpendicular bisector of a and b . $|z - a| = k|z - b|$, $k \neq 1$: a circle (square and collect). $\arg(z - a) = \beta$: half-line from a at angle β . For regions, replace $=$ by $<$ or $>$ and test one convenient point. Under $w = \frac{az + b}{cz + d}$: make z the subject, then take moduli (or arguments) of both sides. A circle through the pole $z = -\frac{d}{c}$ maps to a line; otherwise circle to circle.

Worked example – image of a circle (WFM02 June 2025 Q3). $w = \frac{9i - iz}{1 + z}$ ($z \neq -1$), P on $|z| = 1$. Rearranging: $z(w + i) = 9i - w$, so $z = \frac{9i - w}{w + i}$ and $|z| = 1$ forces $|9i - w| = |w + i|$, i.e. $u^2 + (v - 9)^2 = u^2 + (v + 1)^2$, so $20v = 80$ and the image is the line $v = 4$ – the perpendicular bisector of $9i$ and $-i$. Spot-check: $z = i \mapsto 5 + 4i$, $z = -i \mapsto -5 + 4i$.

2 Practice

2.1 Describe fully the locus given by $|z - 2i| = 3$. [2]

2.2 Describe fully the locus given by $|z - 3| = |z + 1|$. [2]

2.3 Show that the locus $|z - 2| = 2|z + 1|$ is a circle, and state its centre and radius. [4]

2.4 Describe the locus given by $\arg\left(\frac{z - 1}{z + 1}\right) = \frac{\pi}{2}$. [3]

3 Exam-style questions

3.1 (*Original.*) The transformation T from the z -plane to the w -plane is given by $w = \frac{1}{z}$, $z \neq 0$. Show that T maps the circle $|z - 2| = 2$ onto the straight line $\operatorname{Re} w = \frac{1}{4}$, and sketch this line. [5]

3.2 (*Edexcel IAL WFM02 June 2025 Q3.*) The transformation T from the z -plane to the w -plane is given by $w = \frac{9i - iz}{1 + z}$, $z \neq -1$. The point P represents the complex number z , where $|z| = 1$. (a) Sketch the locus of P as z varies. (b) Given that the image under T of the locus of P , where $z \neq -1$, is the line l , determine, in simplest form, a Cartesian equation for l . (c) Write down the image under T of the point $z = i$ and verify that it lies on l . [9]

Solutions

2.1 Circle, centre $2i$ (the point $(0, 2)$), radius 3.

2.2 Perpendicular bisector of the points 3 and -1 : the vertical line $\operatorname{Re} z = 1$.

2.3 With $z = x + iy$: $(x-2)^2 + y^2 = 4[(x+1)^2 + y^2]$, so $x^2 - 4x + 4 + y^2 = 4x^2 + 8x + 4 + 4y^2$, i.e. $3x^2 + 12x + 3y^2 = 0$, so $x^2 + 4x + y^2 = 0$, i.e. $(x+2)^2 + y^2 = 4$: circle, centre $(-2, 0)$, radius 2. (Check: $z = 0$ gives $|-2| = 2|1|$ ✓ and 0 lies on it.)

2.4 $\arg\left(\frac{z-1}{z+1}\right) = \arg(z-1) - \arg(z+1)$ is the angle at z subtended by the segment from -1 to 1 . Equal to $\frac{\pi}{2}$: the semicircle $x^2 + y^2 = 1$, $y > 0$, excluding the endpoints ± 1 .

3.1 $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$. Then $|\frac{1}{w} - 2| = 2$ gives $|1 - 2w| = 2|w|$. Squaring: $1 - 2w - 2\bar{w} + 4|w|^2 = 4|w|^2$, so $w + \bar{w} = \frac{1}{2}$, i.e. $\operatorname{Re} w = \frac{1}{4}$ – a vertical line in the w -plane, $\frac{1}{4}$ unit to the right of the imaginary axis. (Check: $z = 4$ is on the circle, and $z = 4 \mapsto w = \frac{1}{4}$ ✓.)

3.2 (a) Unit circle, centre the origin. (b) $z = \frac{9i - w}{w + i}$, so $|9i - w| = |w + i| \cdot |z| = |w + i|$.

With $w = u + iv$: $9i - w = -u + i(9 - v)$, so $u^2 + (9 - v)^2 = u^2 + (v + 1)^2$, giving

$81 - 18v = 2v + 1$, i.e. $20v = 80$: $v = 4$. (c) $z = i$: $w = \frac{9i - i^2}{1 + i} = \frac{1 + 9i}{1 + i} = \frac{(1 + 9i)(1 - i)}{2} = \frac{10 + 8i}{2} = 5 + 4i$, whose imaginary part is 4, so it lies on l ✓.

Local review guidance: z made the subject before any modulus work; modulus equation expanded in u, v ; line/circle description with geometry named (bisector, centre, radius). This is authored classroom guidance.