5.6 Matrices, transformations, series and induction

Name: _____ Class: _____ Date: _____ **Total: 36 marks**

KEY WORDS determinant • inverse matrix • enlargement • area scale factor • mathematical induction

Matrices, transformations, series and induction

Objective

Carry out 2×2 matrix algebra (determinants, inverses, matrix equations); identify and combine geometrical transformations; use the determinant as an area scale factor; sum finite series with the standard formulae; and complete induction proofs (FP1 5, 6, 7, 8).

1 Key knowledge and worked examples

Inverse and equations. $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$; $\mathbf{M}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. A matrix equation $\mathbf{M}^{-1} = \lambda\mathbf{M} + \mu\mathbf{I}$ is solved by comparing all four entries.

Worked example – entry comparison (WFM01 June 2025 Q1). $\mathbf{M} = \begin{pmatrix} 1 & a \\ 3 & -5 \end{pmatrix}$, $a \neq -\frac{5}{3}$: $\mathbf{M}^{-1} = \dfrac{1}{-(5+3a)}\begin{pmatrix} -5 & -a \\ -3 & 1 \end{pmatrix}$. Given $\mathbf{M}^{-1} = 2\mathbf{M} + 8\mathbf{I} = \begin{pmatrix} 10 & 2a \\ 6 & -2 \end{pmatrix}$, the $(2,1)$ entries give $\dfrac{-3}{-(5+3a)} = 6 \Rightarrow 5 + 3a = \frac{1}{2}$, so $a = -\frac{3}{2}$; the remaining entries agree.

Worked example – reading transformations off matrices (WFM01 June 2025 Q2). $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ sends $\mathbf{i} \to (0,-1)^T$, $\mathbf{j} \to (1,0)^T$: rotation 90° clockwise about O. $\mathbf{Q} = 5\mathbf{I}$: enlargement, factor 5, centre O. $\mathbf{R}$ (reflection in $y = -x$) $= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ maps $(4,3)^T \to (-3,-4)^T = A$. If $\mathbf{Q}$ after $\mathbf{P}$ sends B to A, then $B = \mathbf{P}^{-1}\mathbf{Q}^{-1}A = \frac{1}{5}\mathbf{P}^{-1}(-3,-4)^T = \frac{1}{5}(4,-3)^T = (0.8,-0.6)$.

Worked example – a sum to 2n (WFM01 June 2025 Q8a) and an induction (Q8b). $\sum_{r=1}^{2n}(2r^2-1) = 2\cdot\dfrac{(2n)(2n+1)(4n+1)}{6} - 2n = \dfrac{4n(4n^2+3n-1)}{3} = \frac{4}{3}n(n+1)(4n-1)$ (using $4n^2+3n-1 = (n+1)(4n-1)$). Induction for $\sum_{r=1}^{n} r(3r-2)^2 = \frac{1}{4}n^2(n+1)(9n-7)$: true at $n = 1$; adding the $(k+1)$th term $(k+1)(3k+1)^2$ to $\frac{1}{4}k^2(k+1)(9k-7)$ gives $\frac{1}{4}(k+1)(9k^3+29k^2+24k+4)$, and $(k+1)(k+2)(9k+2)$ expands to the same cubic – the target form for $n = k+1$.

2 Practice

2.1 For $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, find $\mathbf{AB}$, $\det\mathbf{A}$ and $\mathbf{A}^{-1}$. [4]

2.2 Describe fully the single geometrical transformation represented by (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; (b) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$; (c) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$; (d) $\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$. [4]

2.3 A triangle of area 2 is transformed by $\mathbf{M}$ with $\det \mathbf{M} = -7$. Find the area of the image and state what the sign tells you. [2]

2.4 Use standard formulae to show that $\sum_{r=1}^{n}(r+1)^2 = \frac{n}{6}(2n^2 + 9n + 13)$. [4]

2.5 Prove by induction that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for all positive integers n. [3]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WFM01 June 2025 Q1.)* $\mathbf{M} = \begin{pmatrix} 1 & a \\ 2 & -3 \end{pmatrix}$, $a \neq -\frac{3}{2}$. (a) Determine $\mathbf{M}^{-1}$ in terms of a. (b) Given that $\mathbf{M}^{-1} = 2\mathbf{M} + 4\mathbf{I}$, determine the value of a. [6]

3.2 *(Adapted from Edexcel IAL WFM01 June 2025 Q8.)* (a) Using standard summation formulae, show that $\sum_{r=1}^{2n}(2r^2 - 1) = \frac{4}{3}n(n+1)(4n-1)$. (b) Prove by induction that $\sum_{r=1}^{n} r(3r-2)^2 = \frac{1}{4}n^2(n+1)(9n-7)$. (c) Hence solve the equation $8\sum_{r=1}^{n} r(3r-2)^2 = 15\sum_{r=1}^{2n}(2r^2-1)$. [13]

Solutions

2.1 $\mathbf{AB} = \begin{pmatrix} 1 & -3 \\ 4 & -2 \end{pmatrix}$; $\det \mathbf{A} = 10$; $\mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$.

2.2 (a) Reflection in $y = x$; (b) enlargement, scale factor 3, centre O; (c) stretch, factor 2, parallel to the x-axis; (d) rotation 30° anticlockwise about O.

2.3 Image area $= 2 \times 7 = 14$; the negative determinant means the transformation reverses orientation (includes a reflection).

2.4 $\sum(r^2+2r+1) = \frac{n(n+1)(2n+1)}{6} + n(n+1) + n = \frac{n(n+1)(2n+1)+6n(n+1)+6n}{6} = \frac{n(2n^2+3n+1+6n+6+6)}{6} = \frac{n(2n^2+9n+13)}{6}$ ✓ (check $n = 1$: 4 and $\frac{24}{6} = 4$).

2.5 $n = 1$: both sides $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ✓. If true for k: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$ ✓. True for $n = 1$ and $n = k \Rightarrow n = k+1$; by induction, true for all positive integers.

3.1 (a) $\det \mathbf{M} = -3 - 2a$, so $\mathbf{M}^{-1} = \dfrac{1}{-(3+2a)}\begin{pmatrix} -3 & -a \\ -2 & 1 \end{pmatrix}$. (b) $2\mathbf{M} + 4\mathbf{I} = \begin{pmatrix} 6 & 2a \\ 4 & -2 \end{pmatrix}$. Entry $(2,2)$: $\dfrac{1}{-(3+2a)} = -2 \Rightarrow 3 + 2a = \frac{1}{2} \Rightarrow a = -\frac{5}{4}$. Every other entry confirms it: $(2,1)$: $\dfrac{-2}{-1/2} = 4$ ✓; $(1,1)$: $\dfrac{-3}{-1/2} = 6$ ✓; $(1,2)$: $\dfrac{-a}{-1/2} = 2a$ ✓ (identically true).

3.2 (a) $2\sum_1^{2n} r^2 - 2n = \dfrac{2(2n)(2n+1)(4n+1)}{6} - 2n = \dfrac{2n(2n+1)(4n+1) - 6n}{3} = \dfrac{2n(8n^2+6n-2)}{3} = \dfrac{4n(4n^2+3n-1)}{3} = \frac{4}{3}n(n+1)(4n-1)$ (since $4n^2 + 3n - 1 = (n+1)(4n-1)$). (b) Base $n = 1$: $1 = \frac{1}{4} \cdot 1 \cdot 2 \cdot 2$ ✓. Step: adding $(k+1)(3k+1)^2$ to $\frac{1}{4}k^2(k+1)(9k-7)$ gives $\frac{1}{4}(k+1)[9k^3 + 29k^2 + 24k + 4] = \frac{1}{4}(k+1)(k+1)(k+2)(9k+2)$, the target for $n = k+1$; close with the induction statement. (c) $8 \cdot \frac{1}{4}n^2(n+1)(9n-7) = 15 \cdot \frac{4}{3}n(n+1)(4n-1)$ – cancel $n(n+1)$ (positive): $2n(9n-7) = 20(4n-1) \Rightarrow 9n^2 - 47n + 10 = 0 \Rightarrow (9n-2)(n-5) = 0$, so $n = 5$ (n a positive integer).

Local review guidance: inverse formula and entry-by-entry comparison; exact values; induction with base case, inductive step and closing sentence. This is authored classroom guidance.