

4.5 Integration methods and volumes of revolution

Name: _____ Class: _____ Date: _____

Total: 28 marks

KEY WORDS substitution

• integration by parts

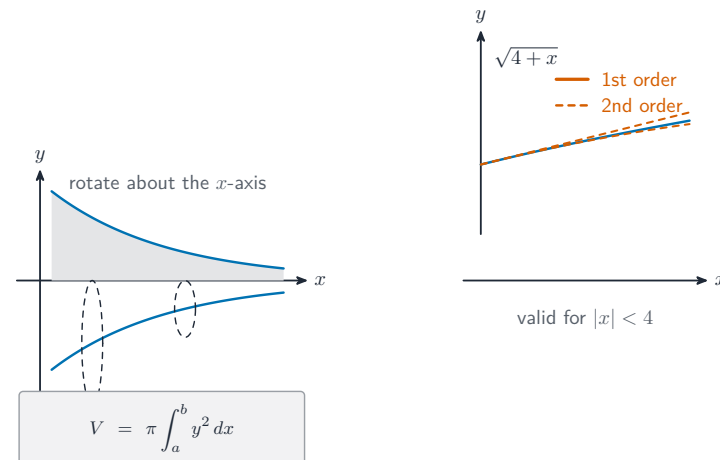
• volume of revolution

Objective

Integrate by substitution (given), by parts (including repeats and $\ln x$), and via partial fractions; and compute volumes of revolution $V = \pi \int y^2 dx$ including parametric curves. Original teaching questions for specification items P4.6.1 to P4.6.3 and P4.6.5; the exam-style part adapts reviewed WMA14 June 2025 demand.

1 Key knowledge and worked examples

- **Substitution** (usually given): convert everything to u (including dx via $\frac{du}{dx}$), integrate, substitute back.
- **Parts** : $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$. Know $\int \ln x dx = x \ln x - x + c$; repeated parts for $\int x^2 e^x dx$.
- **Partial fractions** integrate to logs and powers.
- **Volume of revolution** : $V = \pi \int_a^b y^2 dx$; parametric: $V = \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt} dt$.



Square first; identities handle squared trig.

Worked example – parts that repeats. Find $\int e^x \cos x dx$.

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

$$2 \int e^x \cos x dx = e^x (\sin x + \cos x) \implies \int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + c$$

- Check by differentiating: $\frac{1}{2} e^x (\sin x + \cos x + \cos x - \sin x) = e^x \cos x \checkmark$.

Worked example – volume with an identity. $y = 3 \cos x$ on $[0, \frac{\pi}{2}]$ rotates about the x -axis:

$$V = \pi \int_0^{\pi/2} 9 \cos^2 x dx = \frac{9\pi}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{9\pi^2}{4}$$

- Check: the mean of $\cos^2$ over a half-turn is $\frac{1}{2}$, so $V \approx 9\pi \times \frac{1}{2} \times \frac{\pi}{2} \checkmark$ exactly.

2 Practice

2.1 Use the substitution $u = x^2 + 1$ to find $\int x(x^2 + 1)^3 dx$.

[3]

2.2 Find $\int x e^{2x} dx$ by parts. [3]

2.3 Find $\int \ln 3x dx$. [3]

2.4 Find $\int \frac{4x + 5}{(x + 1)(x + 2)} dx$ using partial fractions. [5]

2.5 Find the volume when $y = \sqrt{x}$ on $[0, 4]$ rotates about the x -axis. [4]

3 Exam-style questions

3.1 (*Adapted from Edexcel IAL WMA14 June 2025 Q7.*) Use the substitution $u = \tan x$ to find $\int \tan x \sec^2 x dx$. [4]

3.2 (*Structure adapted from Edexcel IAL WMA14 June 2025 Q9.*) The region under $y = 2 \sin x$ for $0 \leq x \leq \pi$ rotates about the x -axis. Find the exact volume, showing your use of a double-angle identity. [6]

Solutions

2.1 $du = 2x \, dx$: $\frac{1}{2} \int u^3 du = \frac{(x^2+1)^4}{8} + c$.

2.2 $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$ (parts with $u = x$, $\frac{dv}{dx} = e^{2x}$).

2.3 $x \ln 3x - x + c$ (parts with $u = \ln 3x$, $\frac{dv}{dx} = 1$).

2.4 $\frac{4x+5}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{3}{x+2}$ (check at $x = 0$: LHS $\frac{5}{2}$, RHS $1 + \frac{3}{2}$ ✓):

$$\ln|x+1| + 3\ln|x+2| + c$$

2.5 $V = \pi \int_0^4 x \, dx = \pi \left[\frac{x^2}{2} \right]_0^4 = 8\pi$.

3.1 $u = \tan x$: $\frac{du}{dx} = \sec^2 x$, so $\sec^2 x \, dx = du$:

$$\int \tan x \sec^2 x \, dx = \int u \, du = \frac{u^2}{2} + c = \frac{\tan^2 x}{2} + c$$

• Check: $\frac{d}{dx} \left(\frac{1}{2} \tan^2 x \right) = \tan x \sec^2 x$ ✓.

3.2 $V = \pi \int_0^\pi 4 \sin^2 x \, dx = 4\pi \int_0^\pi \frac{1}{2}(1 - \cos 2x) \, dx = 2\pi \left[x - \frac{\sin 2x}{2} \right]_0^\pi = 2\pi^2$.

• Check: the mean of $\sin^2$ over a full half-turn is $\frac{1}{2}$: $4\pi \times \frac{1}{2} \times \pi = 2\pi^2$ ✓ exactly.

Local review guidance: substitution applied to dx too, or parts with the right u ; exact antiderivatives and volumes; identity used before integrating a squared trig function. This is authored classroom guidance.