4.5 Integration methods and volumes of revolution

Name: _____ Class: _____ Date: _____ **Total: 28 marks**

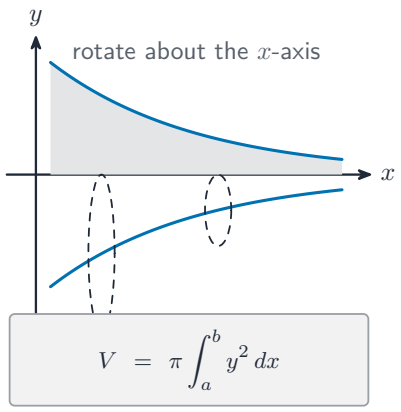
KEY WORDS substitution • integration by parts • volume of revolution

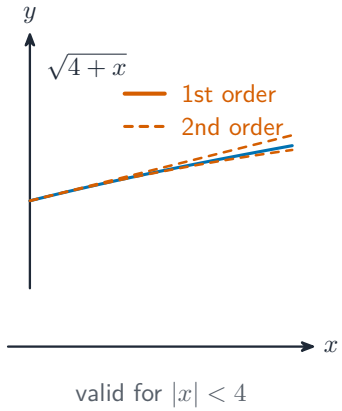
Objective

Integrate by substitution (given), by parts (including repeats and $\ln x$), and via partial fractions; and compute volumes of revolution $V = \pi \int y^2 dx$ including parametric curves. Original teaching questions for specification items P4.6.1 to P4.6.3 and P4.6.5; the exam-style part adapts reviewed WMA14 June 2025 demand.

1 Key knowledge and worked examples

- **Substitution** (usually given): convert everything to u (including dx via $\frac{du}{dx}$), integrate, substitute back.
- **Parts** : $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Know $\int \ln x\, dx = x\ln x - x + c$; repeated parts for $\int x^2 e^x dx$.
- **Partial fractions** integrate to logs and powers.
- **Volume of revolution** : $V = \pi \int_a^b y^2\, dx$; parametric: $V = \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt}\, dt$.





Square first; identities handle squared trig.

Worked example – parts that repeats. Find $\displaystyle\int e^x \cos x\,dx$.

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

$$2\int e^x \cos x\,dx = e^x(\sin x + \cos x) \implies \int e^x \cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + c$$

- Check by differentiating: $\frac{1}{2}e^x(\sin x + \cos x + \cos x - \sin x) = e^x \cos x$ ✓.

Worked example – volume with an identity. $y = 3\cos x$ on $[0, \frac{\pi}{2}]$ rotates about the x-axis:

$$V = \pi \int_0^{\pi/2} 9\cos^2 x\,dx = \frac{9\pi}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{9\pi^2}{4}$$

- Check: the mean of $\cos^2$ over a half-turn is $\frac{1}{2}$, so $V \approx 9\pi \times \frac{1}{2} \times \frac{\pi}{2}$ ✓ exactly.

2 Practice

2.1 Use the substitution $u = x^2 + 1$ to find $\displaystyle\int x(x^2+1)^3\,dx$. [3]

2.2 Find $\int xe^{2x}\,dx$ by parts. [3]

2.3 Find $\int \ln 3x\,dx$. [3]

2.4 Find $\int \frac{4x+5}{(x+1)(x+2)}\,dx$ using partial fractions. [5]

2.5 Find the volume when $y = \sqrt{x}$ on $[0, 4]$ rotates about the x-axis. [4]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA14 June 2025 Q7.)* Use the substitution $u = \tan x$ to find $\displaystyle\int \tan x \sec^2 x \, dx$. [4]

3.2 *(Structure adapted from Edexcel IAL WMA14 June 2025 Q9.)* The region under $y = 2\sin x$ for $0 \leq x \leq \pi$ rotates about the x-axis. Find the exact volume, showing your use of a double-angle identity. [6]

Solutions

2.1 $du = 2x\,dx$: $\frac{1}{2}\int u^3 du = \frac{(x^2+1)^4}{8} + c$.

2.2 $\frac{1}{2}x\mathrm{e}^{2x} - \frac{1}{4}\mathrm{e}^{2x} + c$ (parts with $u = x$, $\frac{dv}{dx} = \mathrm{e}^{2x}$).

2.3 $x\ln 3x - x + c$ (parts with $u = \ln 3x$, $\frac{dv}{dx} = 1$).

2.4 $\dfrac{4x+5}{(x+1)(x+2)} = \dfrac{1}{x+1} + \dfrac{3}{x+2}$ (check at $x = 0$: LHS $\frac{5}{2}$, RHS $1 + \frac{3}{2}$ ✓):

$$\ln|x+1| + 3\ln|x+2| + c$$

2.5 $V = \pi\displaystyle\int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi$.

3.1 $u = \tan x$: $\dfrac{du}{dx} = \sec^2 x$, so $\sec^2 x\,dx = du$:

$$\int \tan x \sec^2 x\,dx = \int u\,du = \frac{u^2}{2} + c = \frac{\tan^2 x}{2} + c$$

- Check: $\dfrac{d}{dx}\left(\frac{1}{2}\tan^2 x\right) = \tan x \sec^2 x$ ✓.

3.2 $V = \pi\displaystyle\int_0^\pi 4\sin^2 x\,dx = 4\pi\int_0^\pi \tfrac{1}{2}(1-\cos 2x)\,dx = 2\pi\left[x - \frac{\sin 2x}{2}\right]_0^\pi = 2\pi^2$.

- Check: the mean of $\sin^2$ over a full half-turn is $\frac{1}{2}$: $4\pi \times \frac{1}{2} \times \pi = 2\pi^2$ ✓ exactly.

Local review guidance: substitution applied to dx too, or parts with the right u; exact antiderivatives and volumes; identity used before integrating a squared trig function. This is authored classroom guidance.