4.3 The binomial series for rational powers

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS binomial series • validity • rational

Objective

Expand $(a+bx)^n$ for any rational n with the validity condition, combine expansions via partial fractions, and use expansions for approximations. Original teaching questions for specification item P4.4.1; the exam-style part adapts reviewed WMA14 June 2025 demand.

1 Key knowledge and worked examples

The **binomial series** for any rational n, valid for $|x| < 1$:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

For $(a+bx)^n = a^n\left(1+\frac{b}{a}x\right)^n$, valid $|x| < \left|\frac{a}{b}\right|$. The series continues indefinitely unless n is a positive integer; reject x values outside the validity. Partial fractions first when the expression is rational.

Worked example (adapted from Edexcel IAL WMA14 June 2025 Q8). Expand $\sqrt{4+x}$ in ascending powers of x to three non-zero terms.

- Factor: $2\left(1+\frac{x}{4}\right)^{1/2}$, valid $|x| < 4$:

$$\sqrt{4+x} = 2\left[1 + \frac{1}{2}\frac{x}{4} + \frac{\frac{1}{2}(-\frac{1}{2})}{2}\left(\frac{x}{4}\right)^2 + \cdots\right] = 2 + \frac{x}{4} - \frac{x^2}{64} + \cdots$$

- Check at $x = 1$: $2 + 0.25 - 0.0156 = 2.2344$ against $\sqrt{5} = 2.2361$ ✓ (2 d.p.).

Worked example – negative n. Expand $(1-2x)^{-2}$ to three terms, $|x| < \frac{1}{2}$.

$$1 + (-2)(-2x) + \frac{(-2)(-3)}{2}(-2x)^2 = 1 + 4x + 12x^2 + \cdots$$

- Check at $x = 0.1$: series $1+0.4+0.12 = 1.52$; true $(0.8)^{-2} = 1.5625$ ✓ (converging).

2 Practice

2.1 Expand $(1+x)^{\frac{1}{3}}$ to three terms. [3]

2.2 Expand $(2-x)^{-1}$ to three terms and state the validity. [3]

2.3 Expand $\sqrt{9+x}$ to three terms. [3]

2.4 Use your answer to 2.3 to estimate $\sqrt{9.4}$ to 4 decimal places. [2]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA14 June 2025 Q8.)*

(a) Find the first three non-zero terms of the binomial series expansion of $\sqrt{4+x}$ in ascending powers of x, giving each coefficient as a simplified fraction.

(b) Write down the first three non-zero terms of the expansion of $\sqrt{4-x}$.

(c) Hence show that $\sqrt{\dfrac{4+x}{4-x}} \approx 1 + \dfrac{x}{4}$, and use it with $x = 1$ to give a rational approximation to one decimal place check of $\sqrt{\frac{5}{3}}$. [7]

3.2 Express $\dfrac{7-x}{(2-x)(1+x)}$ in partial fractions and hence expand it in ascending powers of x to three terms, stating the validity. [7]

Solutions

2.1

$$1 + \tfrac{x}{3} - \tfrac{x^2}{9} + \cdots$$

Coefficient of x^2: $\frac{1}{3}(-\frac{2}{3}) \cdot \frac{1}{2} = -\frac{1}{9}$.

2.2

$$(2-x)^{-1} = \tfrac{1}{2}\left(1 - \tfrac{x}{2}\right)^{-1} = \tfrac{1}{2} + \tfrac{x}{4} + \tfrac{x^2}{8}, \qquad |x| < 2.$$

2.3

$$\sqrt{9+x} = 3\left(1 + \tfrac{x}{9}\right)^{1/2} = 3 + \tfrac{x}{6} - \tfrac{x^2}{216} + \cdots$$

2.4 $x = 0.4$: $3 + 0.0667 - 0.00074 = 3.0659$; true $\sqrt{9.4} = 3.06594$ ✓.

3.1 (a) From the worked example: $2 + \frac{x}{4} - \frac{x^2}{64}$. (b) Replace x by $-x$: $2 - \frac{x}{4} - \frac{x^2}{64}$. (c) Divide (b)'s expansion into (a)'s:

$$\frac{2 + \frac{x}{4} - \frac{x^2}{64} + \cdots}{2 - \frac{x}{4} - \frac{x^2}{64} + \cdots} = \frac{1 + \frac{x}{8} - \frac{x^2}{128}}{1 - \frac{x}{8} - \frac{x^2}{128}} \approx \left(1 + \frac{x}{8}\right)\left(1 + \frac{x}{8}\right) \approx 1 + \frac{x}{4}$$

(to first order; the x^2 terms cancel to this accuracy: $\frac{x}{8} + \frac{x}{8} = \frac{x}{4}$ and $\frac{x^2}{64} - \frac{x^2}{64} - \frac{x^2}{64} = -\frac{x^2}{64}$, an order-2 remainder). With $x = 1$: $\sqrt{\frac{5}{3}} \approx 1.25$; true value $1.2909\ldots$ – a first-order approximation with visible error, improving with smaller x ✓.

3.2 $\dfrac{7-x}{(2-x)(1+x)} = \dfrac{A}{2-x} + \dfrac{B}{1+x}$: $7 - x = A(1+x) + B(2-x)$. At $x = 2$: $5 = 3A \Rightarrow A = \frac{5}{3}$. At $x = -1$: $8 = 3B \Rightarrow B = \frac{8}{3}$.

$$\frac{5}{3}(2-x)^{-1} + \frac{8}{3}(1+x)^{-1} = \frac{5}{6}\left(1 - \frac{x}{2}\right)^{-1} + \frac{8}{3}(1+x)^{-1}$$

$$= \frac{5}{6}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \cdots\right) + \frac{8}{3}(1 - x + x^2 - \cdots) = \frac{7}{2} - \frac{9}{4}x + \frac{23}{8}x^2 + \cdots$$

valid for $|x| < 1$ (the tighter of $|x| < 2$ and $|x| < 1$). Check at $x = 0$: LHS $= \frac{7}{2}$; series $= \frac{5}{6} + \frac{8}{3} = \frac{7}{2}$ ✓.

Local review guidance: correct n and $\frac{b}{a}$ scaling; simplified fractional coefficients; validity stated and approximations used within it. This is authored classroom guidance.