

4.1 Proof by contradiction and partial fractions

Name: _____ Class: _____ Date: _____ Total: 23 marks

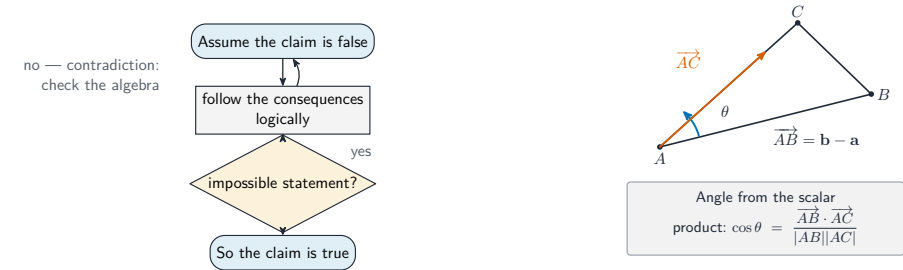
KEY WORDS proof by contradiction • partial fractions • repeated factor
• improper fraction

Objective

Prove statements by contradiction ($\sqrt{2}$ irrational, infinitely many primes, unfamiliar claims) and decompose rational functions into partial fractions, including repeated linear factors and improper fractions. Original teaching questions for specification items P4.1.1 and P4.2.1; the exam-style part adapts reviewed WMA14 June 2025 demand.

1 Key knowledge and worked examples

Proof by contradiction : assume the negation of the claim, argue logically until an impossibility, conclude the claim. **Partial fractions** split a rational function into sums over its denominator’s factors; a repeated factor $(ax + b)^2$ contributes $\frac{B}{ax + b} + \frac{C}{(ax + b)^2}$; improper fractions divide first.



$\sqrt{2}$ irrational and the infinity of primes both run this loop

Name the assumption and the contradiction explicitly.

Worked example – contradiction. Prove there is no smallest positive rational number.

- Suppose q IS the smallest positive rational. Then $\frac{q}{2}$ is positive, rational, and $\frac{q}{2} < q$

– a smaller one exists, contradicting the assumption. ■

Worked example – repeated factor. Decompose $\frac{3x^2 + 7x + 6}{x(x + 1)^2}$.

$$\frac{3x^2 + 7x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \implies 3x^2 + 7x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx$$

- $x = 0$: $A = 6$. $x = -1$: $2 = -C \Rightarrow C = -2$. x^2 coefficients: $3 = A + B \Rightarrow B = -3$.

$$\frac{6}{x} - \frac{3}{x + 1} - \frac{2}{(x + 1)^2}$$

- Check at $x = 1$: LHS = $16/4 = 4$; RHS = $6 - \frac{3}{2} - \frac{1}{2} = 4$ ✓.

2 Practice

2.1 Prove by contradiction that $\sqrt{3}$ is irrational. [4]

2.2 Decompose $\frac{7x + 10}{(x + 1)(x + 3)}$. [3]

2.3 Decompose $\frac{5x + 7}{(x - 1)^2}$. [3]

2.4 Write $\frac{x^2 + 2x + 5}{x^2 + x}$ in the form $1 + \frac{A}{x} + \frac{B}{x + 1}$. [4]

3 Exam-style questions

3.1 (*Adapted from Edexcel IAL WMA14 June 2025 Q10.*) Use proof by contradiction to show that $|\sin x + \cos x| < 1$ for all angles x with $90^\circ < x < 180^\circ$. [4]

3.2 Decompose $\frac{2x^2 + 5x + 4}{(x + 2)(x + 1)^2}$ and verify your decomposition at $x = 1$. [5]

Solutions

2.1 Suppose $\sqrt{3} = \frac{p}{q}$ in lowest terms. Then $3q^2 = p^2$, so p^2 is divisible by 3, so p is (if $p = 3m$, $p^2 = 9m^2$; conversely a prime dividing p^2 divides p). Write $p = 3m$: $3q^2 = 9m^2 \Rightarrow q^2 = 3m^2$, so q is divisible by 3 too – both divisible contradicts lowest terms. ■

2.2 $7x + 10 = A(x + 3) + B(x + 1)$: at $x = -1$, $A = \frac{3}{2}$; at $x = -3$, $B = \frac{11}{2}$. (Or compare coefficients: $A + B = 7$, $3A + B = 10$.)

2.3 With $\frac{5x + 7}{(x - 1)^2} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2}$, multiply through: $5x + 7 = A(x - 1) + B$. Comparing x -coefficients, $A = 5$; comparing constants, $7 = -5 + B \Rightarrow B = 12$.

2.4 $\frac{x^2 + 2x + 5}{x^2 + x} = 1 + \frac{x + 5}{x^2 + x} = 1 + \frac{x + 5}{x(x + 1)}$; decompose $\frac{x + 5}{x(x + 1)} = \frac{5}{x} - \frac{4}{x + 1}$. So $A = 5$, $B = -4$. Check at $x = 1$: $1 + 5 - 2 = 4$; original $\frac{8}{2} = 4$ ✓.

3.1 Suppose some x in $90^\circ < x < 180^\circ$ has $|\sin x + \cos x| \geq 1$. Squaring: $1 + \sin 2x \geq 1$, so $\sin 2x \geq 0$. But $180^\circ < 2x < 360^\circ$ forces $\sin 2x < 0$ – a contradiction. So $|\sin x + \cos x| < 1$ throughout. ■ Check at $x = 135^\circ$: $|0| = 0 < 1$ ✓.

3.2 $\frac{2x^2 + 5x + 4}{(x + 2)(x + 1)^2} = \frac{A}{x + 2} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$: $2x^2 + 5x + 4 = A(x + 1)^2 + B(x + 2)(x + 1) + C(x + 2)$.

• $x = -2$: $8 - 10 + 4 = 2 = A$. $x = -1$: $2 - 5 + 4 = 1 = C$. x^2 : $2 = A + B \Rightarrow B = 0$.

$$\frac{2}{x + 2} + \frac{1}{(x + 1)^2}$$

Check at $x = 1$: LHS = $\frac{2+5+4}{(1+2)(1+1)^2} = \frac{11}{12}$; RHS = $\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$ ✓.

Local review guidance: assumption of the negation stated, or correct partial-fraction identity set up; exact values; verification or the named contradiction. This is authored classroom guidance.