2.8 Stationary points

Name: _____ Class: _____ Date: _____ **Total: 36 marks**

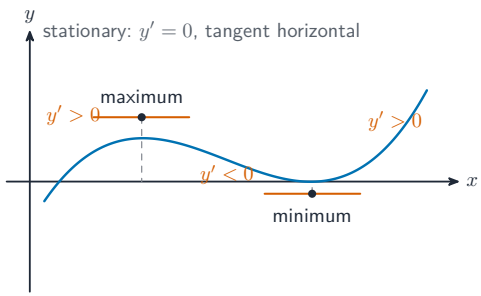
KEY WORDS stationary point • maximum • minimum • increasing • decreasing

Objective

Find stationary points from $\frac{dy}{dx} = 0$, classify them with the derivative's sign or the second derivative, identify increasing and decreasing stretches, and solve practical maximum/minimum problems. Original teaching questions for specification item P2.7.1; the exam-style part adapts reviewed WMA12 June 2025 demand.

1 Key knowledge and worked examples

- **Stationary points** : $\frac{dy}{dx} = 0$. Tangent horizontal there.
- Classify: $f''(x) > 0 \to$ **minimum** ; $f''(x) < 0 \to$ **maximum** ; or read the sign change of $f'(x)$ from $+$ to $-$ (maximum) or $-$ to $+$ (minimum).
- **Increasing** where $f'(x) > 0$; **decreasing** where $f'(x) < 0$.
- Practical problems: write the quantity to optimise in one variable first, using a constraint (volume, length) to eliminate the other variable.



Stationary means the tangent is horizontal; the sign change classifies it.

Worked example. For $y = x^3 - 12x + 1$, find and classify the stationary points.

- $\frac{dy}{dx} = 3x^2 - 12 = 3(x-2)(x+2) = 0 \implies x = \pm 2$.
- $y(2) = 8 - 24 + 1 = -15$; $y(-2) = -8 + 24 + 1 = 17$.
- $\frac{d^2y}{dx^2} = 6x$: at $x = 2$, $12 > 0 \to$ minimum at $(2, -15)$; at $x = -2$, $-12 < 0 \to$ maximum at $(-2, 17)$.
- Check: at $x = 0$ (between), $y' = -12 < 0$ (decreasing) – consistent with max at -2, min at 2 ✓.

Worked example – modelling minimum. An open box has a square base of side x and height h, volume $32\ \text{m}^3$, and no top. Its surface area is $S = x^2 + 4xh$. Show the minimum S is $48\ \text{m}^2$.

- Constraint: $x^2h = 32$, so $h = \frac{32}{x^2}$ and

$$S = x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x}$$

- $\frac{dS}{dx} = 2x - \frac{128}{x^2} = 0 \implies 2x^3 = 128 \implies x = 4$.
- $S = 16 + \frac{128}{4} = 16 + 32 = 48\ \text{m}^2$; $\frac{d^2S}{dx^2} = \frac{256}{x^3} = 4 > 0 \to$ minimum ✓.

2 Practice

2.1 Find the stationary point of $y = x^2 - 8x + 3$ and state its nature. [3]

2.2 Find the stationary points of $y = x^4 - 2x^2$ and classify each. [5]

2.3 For $y = x^3 - 3x^2$, state where the function is increasing and where decreasing. [3]

2.4 For $y = x + \dfrac{1}{x}$ with $x > 0$, find the stationary point and its nature. [4]

2.5 Find the stationary points of $y = 2x^3 + 9x^2 - 24x$. [4]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA12 June 2025 Q9.)* A closed solid has a square cross-section of side x metres and length L metres, with volume 20 m^3. Its total surface area is $S = 2x^2 + 4xL$.

(a) Show that $S = 2x^2 + \dfrac{80}{x}$.

(b) Use calculus to find the value of x that minimises S, and find the minimum S to one decimal place.

(c) Justify that your value is a minimum. [10]

3.2 For $y = x^3 - 6x^2 + 9x$, find the stationary points, determine their nature, and sketch the curve. [7]

Solutions

2.1 $y' = 2x - 8 = 0 \implies x = 4$; $y(4) = -13$; $y'' = 2 > 0$: minimum at $(4, -13)$.

2.2 $y' = 4x^3 - 4x = 4x(x-1)(x+1) = 0$: $x = 0, \pm 1$. $y(\pm 1) = 1 - 2 = -1$; $y(0) = 0$. $y'' = 12x^2 - 4$: at ± 1, $8 > 0 \to$ minima $(\pm 1, -1)$; at 0, $-4 < 0 \to$ maximum $(0, 0)$.

2.3 $y' = 3x^2 - 6x = 3x(x-2)$: increasing for $x < 0$ or $x > 2$; decreasing for $0 < x < 2$.

2.4 $y' = 1 - \dfrac{1}{x^2} = 0 \implies x = 1$ (given $x > 0$). $y(1) = 2$; $y'' = \dfrac{2}{x^3} > 0$: minimum at $(1, 2)$.

2.5 $y' = 6x^2 + 18x - 24 = 6(x+4)(x-1) = 0$: $x = 1$ and $x = -4$. $y(1) = 2 + 9 - 24 = -13$; $y(-4) = -128 + 144 + 96 = 112$. Points $(1, -13)$ and $(-4, 112)$.

3.1 (a) Volume: $x^2 L = 20$, so $L = \dfrac{20}{x^2}$ and

$$S = 2x^2 + 4x \cdot \frac{20}{x^2} = 2x^2 + \frac{80}{x}$$

(b) $\dfrac{dS}{dx} = 4x - \dfrac{80}{x^2} = 0 \implies 4x^3 = 80 \implies x = \sqrt[3]{20} = 2.71$ (3 s.f.).

$$S = 2(20)^{2/3} + \frac{80}{(20)^{1/3}} = 14.74 + 29.48 = 44.2 \text{ m}^2 \text{ (1 d.p.)}$$

(c) $\dfrac{d^2S}{dx^2} = \dfrac{160}{x^3} > 0$ for all $x > 0$ (at $x = \sqrt[3]{20}$ it equals $\frac{160}{20} = 8 > 0$), so the stationary point is a minimum ✓.

3.2 $y' = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$: $x = 1, 3$. $y(1) = 1 - 6 + 9 = 4$; $y(3) = 27 - 54 + 27 = 0$. $y'' = 6x - 12$: at 1, $-6 < 0 \to$ maximum $(1, 4)$; at 3, $6 > 0 \to$ minimum $(3, 0)$. Sketch: crosses the x-axis at $x = 0$ and touches at $x = 3$ (minimum on the axis), rises to $(1, 4)$, falls to $(3, 0)$, then increases – a positive cubic through the origin. Check: $y(0) = 0$ ✓.

Local review guidance: derivative set to zero after eliminating the second variable; exact coordinates; nature justified by second derivative or sign change. This is authored classroom guidance.