

## 2.1 Proof: exhaustion and counterexample

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

**Total: 20 marks**

**KEY WORDS** proof • proof by exhaustion • counterexample

### Objective

Structure mathematical proofs, use proof by exhaustion over complete sets of cases, and disprove statements with a counterexample. Original teaching questions for specification items P2.1.1 to P2.1.3; the exam-style part adapts reviewed WMA12 June 2025 demand.

### 1 Key knowledge and worked examples

A **proof** starts from given assumptions, moves by logical steps, and ends with the target conclusion. Quote every assumption you use; do not assume what you are proving.

- **Proof by exhaustion** lists every possible case and checks each one. The case list must be complete – “every integer is of the form  $3q$ ,  $3q + 1$  or  $3q + 2$ ” is complete; “the first few integers” is not.
- **Disproof by counterexample** needs exactly one example that breaks the claim.

**Worked example – exhaustion.** Prove that for every integer  $n$ , the number  $n^2 + n$  is even.

- The cases split on parity, and every integer is in one case:
- $n$  even,  $n = 2k$ :  $n^2 + n = 4k^2 + 2k = 2(2k^2 + k)$  – even.
- $n$  odd,  $n = 2k + 1$ :  $n^2 + n = (2k + 1)(2k + 2)$ , and  $2k + 2$  is even, so the product is even.
- Both cases give an even number, so  $n^2 + n$  is always even. ■
- Check:  $n = 4$ : 20 ✓;  $n = 5$ : 30 ✓;  $n = -3$ : 6 ✓.

**Worked example – counterexample.** Disprove: “ $2^n + 1$  is prime for every positive integer  $n$ .”

- Try small values: 3, 5, 9, ... – and  $2^3 + 1 = 9 = 3 \times 3$  is not prime.
- One counterexample ( $n = 3$ ) disproves the claim for all  $n$ . ■

## 2 Practice

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**2.1** Disprove by counterexample: " $n^2 + 1$  is prime for all positive integers  $n$ ." [2]

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**2.2** Prove that the sum of three consecutive integers is divisible by 3. [3]

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**2.3** Prove by exhaustion that  $n^2 - 1$  is divisible by 3 for every positive integer  $n$  that is not a multiple of 3. [4]

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**2.4** Prove that the product of two odd integers is odd. [3]

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## 3 Exam-style questions

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**3.1** (*Adapted from Edexcel IAL WMA12 June 2025 Q10.*) Use algebra to prove by exhaustion that for all positive integers  $m$  that are not multiples of 3, the value of  $m^2 + 3m + 2$  is always a multiple of 3. [4]

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**3.2** Prove by exhaustion that no perfect square ends in the digit 8.

[4]

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## Solutions

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**2.1**  $n = 3$ :  $9 + 1 = 10 = 2 \times 5$ , which is not prime. One counterexample disproves the claim.

**2.2** Let the integers be  $(n - 1) + n + (n + 1) = 3n$ , which is divisible by 3 for every integer  $n$ . ■

**2.3** Non-multiples of 3 are  $n = 3q + 1$  or  $n = 3q + 2$  (complete case list).

- $n = 3q + 1$ :  $n^2 - 1 = 9q^2 + 6q = 3(3q^2 + 2q)$  ✓
- $n = 3q + 2$ :  $n^2 - 1 = 9q^2 + 12q + 3 = 3(3q^2 + 4q + 1)$  ✓ Both cases give a multiple of 3. ■

**2.4** Let the integers be  $2a + 1$  and  $2b + 1$ : their product is  $4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$ , which is one more than an even number – odd. ■

**3.1**  $m^2 + 3m + 2 = (m + 1)(m + 2)$ . Cases:  $m = 3q + 1$  or  $m = 3q + 2$ .

- $m = 3q + 1$ :  $(3q + 2)(3q + 3) = 3(3q + 2)(q + 1)$  – multiple of 3.
- $m = 3q + 2$ :  $(3q + 3)(3q + 4) = 3(q + 1)(3q + 4)$  – multiple of 3. Exhausting all non-multiples of 3, the expression is always a multiple of 3. ■ Check:  $m = 1$ : 6;  $m = 2$ : 12;  $m = 4$ : 30;  $m = 5$ : 42 – all divisible by 3 ✓.

**3.2** A square's last digit depends only on its root's last digit. Exhaust the ten endings:

$$0^2 \rightarrow 0, 1^2 \rightarrow 1, 2^2 \rightarrow 4, 3^2 \rightarrow 9, 4^2 \rightarrow 6, 5^2 \rightarrow 5, 6^2 \rightarrow 6, 7^2 \rightarrow 9, 8^2 \rightarrow 4, 9^2 \rightarrow 1$$

The possible endings are 0, 1, 4, 5, 6, 9 – never 8. ■

Local review guidance: complete case structure or a valid counterexample; algebra carried correctly; the conclusion states what was proved, with the case list justified. This is authored classroom guidance.