

Exercise walk-through

7.6

Vectors and further matrix algebra

Pearson Edexcel · International A-Level · Pure Mathematics

7. Integrating hyperbolic and inverse functions

Integrating

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh} \frac{x}{a} + c, \quad \int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh} \frac{x}{a} + c, \quad \int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + c,$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c. \text{ Complete the square first: } 4x^2 + 8x + 9 = 4((x+1)^2 + \frac{5}{4}).$$

Inverse functions: parts with u the inverse, e.g.

$$\int \operatorname{arcosh} 3x \, dx = x \operatorname{arcosh} 3x - \frac{1}{3} \sqrt{9x^2 - 1} + c.$$

10. Further matrix algebra

Matrices and eigenvalues

3×3 det by expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$; singular when $\det = 0$. Eigen check row

by row: components of $\mathbf{M}\mathbf{v}$ equal λv_j . For $\begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ with eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$:

$\lambda = 3$, $a = 2$, $b = -1$, others 1 and -2 . Symmetric matrices: orthogonal $\mathbf{P}$ of normalised eigenvectors, $\mathbf{P}^T \mathbf{A} \mathbf{P}$ diagonal.

Check yourself

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Derive a cosh identity from exponentials; solve a mixed equation as a quadratic in e^x ; state and prove an arsinh log form; place foci and directrices of both conics; tangent by parametric derivative and by discriminant; differentiate and integrate hyperbolic and inverse functions; build and iterate a reduction formula; arc length and surface exactly; cross and triple products; plane forms and skew distance; 3×3 inverse; eigenvalues, eigenvectors, diagonalisation.

Exercises 2.1·2.2

2.1 Find $(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$.

[2]

2.2 Find the volume of the parallelepiped spanned by $\mathbf{a} = (1, 0, 2)$, $\mathbf{b} = (3, 1, 0)$ and $\mathbf{c} = (0, 2, 1)$.

[3]

Exercises 2.1·2.2

2.1 Find $(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$. [2]

2.1 $\mathbf{i}((-3)(-2) - (1)(4)) - \mathbf{j}((2)(-2) - (1)(1)) + \mathbf{k}((2)(4) - (-3)(1)) = 2\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}.$

2.2 Find the volume of the parallelepiped spanned by $\mathbf{a} = (1, 0, 2)$, $\mathbf{b} = (3, 1, 0)$ and $\mathbf{c} = (0, 2, 1)$. [3]

2.2 $\mathbf{b} \times \mathbf{c} = (1 \cdot 1 - 0 \cdot 2, 0 \cdot 0 - 3 \cdot 1, 3 \cdot 2 - 1 \cdot 0) = (1, -3, 6); \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 1 + 0 + 12 = 13.$
Volume 13 cubic units.

Exercises 2.3·2.4

2.3 Evaluate the determinant of $\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 5 & 2 & -2 \end{pmatrix}$. [3]

2.4 Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and write down an orthogonal matrix **P** that diagonalises it. [4]

Exercises 2.3·2.4

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2.3 Expanding along the first row: $2((-1)(-2) - 4 \cdot 2) - 1(0 \cdot (-2) - 4 \cdot 5) + 3(0 \cdot 2 - (-1) \cdot 5) = 2(-6) + 20 + 15 = 23$.

2.4 Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and write down an orthogonal matrix $\mathbf{P}$ that diagonalises it. [4]

2.4 $\det \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix} = (2 - \lambda)^2 - 1 = 0$ gives $\lambda = 3$ or 1 . $\lambda = 3$: $(\mathbf{M} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$ gives $v_1 = v_2$, so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda = 1$: $v_1 = -v_2$, so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Normalising, $\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (or with columns swapped/negated), and $\mathbf{P}^T \mathbf{A} \mathbf{P} = \text{diag}(3, 1)$.

Exercise 3.1

3.1 (*Edexcel IAL WFM03 June 2025 Q8, condensed.*) The plane Π_1 has equation $\mathbf{r} = (4, 2, -1) + \lambda(7, 5, -6) + \mu(-3, 1, 2)$. (a) Determine $(7\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}) \times (-3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. (b) Hence show that the equation of Π_1 can be written as $\mathbf{r} \cdot (8\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}) = p$, stating the value of p . Given that the line of intersection of Π_1 with the plane $\Pi_2 : x - y + z = 7$ has direction $(13, 3, -10)$ and passes through $(\frac{13}{5}, -\frac{17}{5}, 1)$, and that the line l_2 passes through $A(2, 1, 3)$ and $B(3, 0, 2)$, (c) determine the shortest distance between the two lines. [7]

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3.1 (a) $(7, 5, -6) \times (-3, 1, 2) = (5 \cdot 2 - (-6)(1), (-6)(-3) - 7 \cdot 2, 7 \cdot 1 - 5 \cdot (-3)) = (16, 4, 22) = 2(8, 2, 11)$. (b) The normal is $(8, 2, 11)$; substituting the anchor point:

Full solution 3.1

3.1 (a) $(7, 5, -6) \times (-3, 1, 2) = (5 \cdot 2 - (-6)(1), (-6)(-3) - 7 \cdot 2, 7 \cdot 1 - 5 \cdot (-3)) = (16, 4, 22) = 2(8, 2, 11)$. (b) The normal is $(8, 2, 11)$; substituting the anchor point: $p = 4 \cdot 8 + 2 \cdot 2 + (-1) \cdot 11 = 32 + 4 - 11 = 25$. (c) l_2 has direction $\mathbf{b}_2 = (3, 0, 2) - (2, 1, 3) = (1, -1, -1)$. Common normal: $(13, 3, -10) \times (1, -1, -1) = (3 \cdot (-1) - (-10)(-1), (-10)(1) - 13 \cdot (-1), 13 \cdot (-1) - 3 \cdot 1) = (-13, 3, -16)$, with $|(-13, 3, -16)| = \sqrt{434}$. Offset between anchors: $(2, 1, 3) - (\frac{13}{5}, -\frac{17}{5}, 1) = (-\frac{3}{5}, \frac{22}{5}, 2)$. Distance = $\frac{|\frac{39}{5} + \frac{66}{5} - 2|}{\sqrt{434}} = \frac{11}{\sqrt{434}}$.

Exercise 3.2

3.2 (*Edexcel IAL WFM03 June 2025 Q3, condensed.*) $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$, where a and b are constants, has eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. (a) Determine the corresponding eigenvalue. (b) Hence determine a and b . (c) Given that the other eigenvalues of $\mathbf{M}$ are 1 and -2 , determine an eigenvector corresponding to $\lambda = 1$. [6]

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3.2 (a) Row 1: $\mathbf{M}\mathbf{v}$'s first component is $1 - 2 + 4 = 3$, and $3 = \lambda \cdot 1$, so $\lambda = 3$. (b) Row 2: $3 + 2a + b = 3 \times 2 = 6$; row 3: $a + 2 + b = 3 \times 1 = 3$. Subtracting: $a = 2$; then $b = -1$. (c) With $a = 2$, $b = -1$: $(\mathbf{M} - \mathbf{I})\mathbf{v} = \mathbf{0}$ has rows $(0, -1, 4)$ and $(3, 1, -1)$ (and $(2, 1, -2)$). From row 1: $v_2 = 4v_3$; row 2: $3v_1 + v_2 - v_3 = 0$, so with $v_2 = 4v_3$, $3v_1 = -3v_3$ and $v_1 = -v_3$. Taking $v_3 = 1$: eigenvector $(-1, 4, 1)$.

Local review guidance: cross products with the correct cyclic signs (middle term negated); plane constant from an anchor point; eigen checks compare component-wise with λv_j ; distances projected onto the common normal with exact surds. This is authored classroom guidance.