

Exercise walk-through

7.5

Reduction formulae, arc length and surfaces of revolution

Pearson Edexcel · International A-Level · Pure Mathematics

7. Integrating hyperbolic and inverse functions

Integrating

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh} \frac{x}{a} + c, \quad \int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh} \frac{x}{a} + c, \quad \int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + c,$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c. \text{ Complete the square first: } 4x^2 + 8x + 9 = 4\left((x+1)^2 + \frac{5}{4}\right).$$

Inverse functions: parts with u the inverse, e.g.

$$\int \operatorname{arcosh} 3x \, dx = x \operatorname{arcosh} 3x - \frac{1}{3} \sqrt{9x^2 - 1} + c.$$

10. Further matrix algebra

Matrices and eigenvalues

3×3 det by expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$; singular when $\det = 0$. Eigen check row

by row: components of $\mathbf{M}\mathbf{v}$ equal λv_j . For $\begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ with eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$:

$\lambda = 3$, $a = 2$, $b = -1$, others 1 and -2 . Symmetric matrices: orthogonal $\mathbf{P}$ of normalised eigenvectors, $\mathbf{P}^T \mathbf{A} \mathbf{P}$ diagonal.

Check yourself

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Derive a cosh identity from exponentials; solve a mixed equation as a quadratic in e^x ; state and prove an arsinh log form; place foci and directrices of both conics; tangent by parametric derivative and by discriminant; differentiate and integrate hyperbolic and inverse functions; build and iterate a reduction formula; arc length and surface exactly; cross and triple products; plane forms and skew distance; 3×3 inverse; eigenvalues, eigenvectors, diagonalisation.

Exercises 2.1·2.2

2.1 Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$, giving an exact answer.[4]

2.2 The curve $y = \cosh x$, $0 \leq x \leq \ln 2$, is rotated through 2π radians about the x -axis. Find, exactly, the area of the curved surface generated. [4]

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2.1 Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$, giving an exact answer.[4]

$$\mathbf{2.1} \quad \frac{dy}{dx} = \frac{3}{2}x^{1/2}, \text{ so with } u = 1 + \frac{9}{4}x, \quad du = \frac{9}{4}dx: \quad s = \int_0^4 \sqrt{1 + \frac{9}{4}x} \, dx = \frac{4}{9} \left[\frac{2}{3}u^{3/2} \right]_1^{10} = \frac{8}{27} (10^{3/2} - 1).$$

2.2 The curve $y = \cosh x$, $0 \leq x \leq \ln 2$, is rotated through 2π radians about the x -axis. Find, exactly, the area of the curved surface generated. [4]

2.2 $\frac{dy}{dx} = \sinh x$ and $1 + \sinh^2 x = \cosh^2 x$, so

$$S = 2\pi \int_0^{\ln 2} \cosh^2 x \, dx = 2\pi \left[\frac{1}{2} \sinh x \cosh x + \frac{x}{2} \right]_0^{\ln 2}$$

(from $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$). With $\sinh(\ln 2) = \frac{3}{4}$, $\cosh(\ln 2) = \frac{5}{4}$: $S = 2\pi \left(\frac{15}{32} + \frac{1}{2} \ln 2 \right)$,
i.e. $\frac{15\pi}{16} + \pi \ln 2$.

Exercise 2.3

2.3 Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$. (a) Show that, for $n \geq 2$, $nI_n = (n-1)I_{n-2}$. (b) Hence find I_4 . [5]

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2.3 (a) Parts with $u = \sin^{n-1} x$, $dv = \sin x \, dx$: $I_n = (n-1) \int_0^{\pi/2} \cos^2 x \sin^{n-2} x \, dx = (n-1)(I_{n-2} - I_n)$ (the boundary term vanishes at both limits), so $nI_n = (n-1)I_{n-2}$. (b) $I_0 = \frac{\pi}{2}$, $I_2 = \frac{1}{2}I_0 = \frac{\pi}{4}$, $I_4 = \frac{3}{4}I_2 = \frac{3\pi}{16}$.

Exercise 3.1

3.1 (*Edexcel IAL WFM03 June 2025 Q5.*) $I_n = \int_1^6 x^n(3x - 2)^{-1/2} dx$, $n \geq 0$. (a) Show that, for $n \geq 1$, $(3 + 6n)I_n = 4nI_{n-1} + 8 \times 6^n - 2$. (b) Use the reduction formula in part (a) to determine the exact value of $\int_1^6 x^3(3x - 2)^{-1/2} dx$. [9]

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3.1 (a) $v = \frac{2}{3}\sqrt{3x-2}$ (check by differentiating). Parts:

$$I_n = \left[\frac{2}{3}x^n\sqrt{3x-2} \right]_1^6 - \frac{2n}{3} \int_1^6 x^{n-1}\sqrt{3x-2} dx$$

The boundary term: $\frac{2}{3}(6^n \cdot 4 - 1) = \frac{8 \cdot 6^n - 2}{3}$. The integral: $\int_1^6 \frac{x^{n-1}(3x-2)}{\sqrt{3x-2}} dx = 3I_n - 2I_{n-1}$. So $I_n = \frac{8 \cdot 6^n - 2}{3} - \frac{2n}{3}(3I_n - 2I_{n-1})$, which rearranges to $(3+6n)I_n = 4nI_{n-1} + 8 \cdot 6^n - 2$.
 (b) $I_0 = \frac{2}{3}[\sqrt{3x-2}]_1^6 = \frac{2}{3}(4-1) = 2$; $n=1$: $9I_1 = 4(2) + 48 - 2 = 54$, so $I_1 = 6$; $n=2$: $15I_2 = 8(6) + 8(36) - 2 = 334$, so $I_2 = \frac{334}{15}$; $n=3$: $21I_3 = 12 \cdot \frac{334}{15} + 8(216) - 2 = \frac{1336}{5} + 1726 = \frac{9966}{5}$, so $I_3 = \frac{9966}{105} = \frac{3322}{35}$.

Exercise 3.2

3.2 (*Edexcel IAL WFM03 June 2025 Q7a.*) The curve with equation $y = \cos 2x$, $0 \leq x \leq \frac{\pi}{4}$, is rotated through 2π radians about the x -axis. Show that the area of the curved surface generated is $S = 2\pi \int_0^{\pi/4} \cos 2x \sqrt{1 + 4 \sin^2 2x} \, dx$. [3]

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3.2 (*Edexcel IAL WFM03 June 2025 Q7a.*) The curve with equation $y = \cos 2x$, $0 \leq x \leq \frac{\pi}{4}$, is rotated through 2π radians about the x -axis. Show that the area of the curved surface generated is $S = 2\pi \int_0^{\pi/4} \cos 2x \sqrt{1 + 4 \sin^2 2x} dx$. [3]

3.2 $\frac{dy}{dx} = -2 \sin 2x$, so $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 4 \sin^2 2x}$ and $S = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_0^{\pi/4} \cos 2x \sqrt{1 + 4 \sin^2 2x} dx$ (no modulus needed: $\cos 2x \geq 0$ on $[0, \frac{\pi}{4}]$).

Local review guidance: parts orientation chosen so the leftover integral re-expresses in I_n, I_{n-1} ; boundary values at both limits; surface integrals carry the y factor and the surd from $\frac{dy}{dx}$, with exact evaluations. This is authored classroom guidance.