

Exercise walk-through

7.4

Integrating hyperbolic and inverse functions

Pearson Edexcel · International A-Level · Pure Mathematics

7. Integrating hyperbolic and inverse functions

Integrating

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh} \frac{x}{a} + c, \quad \int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh} \frac{x}{a} + c, \quad \int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + c,$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c. \text{ Complete the square first: } 4x^2 + 8x + 9 = 4\left((x+1)^2 + \frac{5}{4}\right).$$

Inverse functions: parts with u the inverse, e.g.

$$\int \operatorname{arcosh} 3x \, dx = x \operatorname{arcosh} 3x - \frac{1}{3} \sqrt{9x^2 - 1} + c.$$

10. Further matrix algebra

Matrices and eigenvalues

3×3 det by expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$; singular when $\det = 0$. Eigen check row

by row: components of $\mathbf{M}\mathbf{v}$ equal λv_j . For $\begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ with eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$:

$\lambda = 3$, $a = 2$, $b = -1$, others 1 and -2 . Symmetric matrices: orthogonal $\mathbf{P}$ of normalised eigenvectors, $\mathbf{P}^T \mathbf{A} \mathbf{P}$ diagonal.

Check yourself

Check yourself

Derive a cosh identity from exponentials; solve a mixed equation as a quadratic in e^x ; state and prove an arsinh log form; place foci and directrices of both conics; tangent by parametric derivative and by discriminant; differentiate and integrate hyperbolic and inverse functions; build and iterate a reduction formula; arc length and surface exactly; cross and triple products; plane forms and skew distance; 3×3 inverse; eigenvalues, eigenvectors, diagonalisation.

Exercises 2.1·2.3

2.1 Find $\int \sinh^3 x \cosh x \, dx$. [3]

2.2 Evaluate $\int_0^1 \frac{dx}{x^2 + 4}$. [4]

2.3 Find $\int \frac{dx}{x^2 - 9}$. [4]

Exercises 2.1·2.3

2.1 Find $\int \sinh^3 x \cosh x \, dx$. [3]

2.1 With $u = \sinh x$: $\frac{1}{4} \sinh^4 x + c$.

2.2 Evaluate $\int_0^1 \frac{dx}{x^2 + 4}$. [4]

2.2 $\left[\frac{1}{9} \arctan \frac{x}{9}\right]_0^1 = \frac{1}{9} \arctan \frac{1}{9}$.

2.3 Find $\int \frac{dx}{x^2 - 9}$. [4]

2.3 $\frac{1}{2 \cdot 3} \ln \left| \frac{x - 3}{x + 3} \right| + c = \frac{1}{6} \ln \left| \frac{x - 3}{x + 3} \right| + c$.

Exercises 2.4·3.1

2.4 Find $\int \frac{dx}{\sqrt{x^2 + 9}}$. [3]

3.1 (*Edexcel IAL WFM03 June 2025 Q2.*) Determine

(i) $\int \frac{dx}{\sqrt{4x^2 + 8x + 9}}$

(ii) $\int \operatorname{arcosh} 3x \, dx$. [7]

Exercises 2.4·3.1

2.4 Find $\int \frac{dx}{\sqrt{x^2 + 9}}$. [3]

2.4 $\operatorname{arsinh} \frac{x}{3} + c$ ($= \ln(x + \sqrt{x^2 + 9}) + c$).

3.1 (*Edexcel IAL WFM03 June 2025 Q2.*) Determine

(i) $\int \frac{dx}{\sqrt{4x^2 + 8x + 9}}$

(ii) $\int \operatorname{arcosh} 3x \, dx$. [7]

3.1 (i) $4x^2 + 8x + 9 = (2x + 2)^2 + 5$, so with $u = 2x + 2$, $dx = \frac{du}{2}$: $\frac{1}{2} \int \frac{du}{\sqrt{u^2 + 5}} = \frac{1}{2} \operatorname{arsinh} \frac{2x+2}{\sqrt{5}} + c$. (ii) By parts, $u = \operatorname{arcosh} 3x$, $dv = dx$: $x \operatorname{arcosh} 3x - \int x \cdot \frac{3}{\sqrt{9x^2 - 1}} dx = x \operatorname{arcosh} 3x - \frac{1}{3} \sqrt{9x^2 - 1} + c$ (because $\frac{d}{dx} \sqrt{9x^2 - 1} = \frac{9x}{\sqrt{9x^2 - 1}}$).

Exercise 3.2

3.2 (*Original.*) Evaluate $\int_0^1 \operatorname{arsinh} x \, dx$, giving your answer in exact form. [4]

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3.2 By parts: $[x \operatorname{arsinh} x]_0^1 - \int_0^1 \frac{x \, dx}{\sqrt{1+x^2}} = \ln(1+\sqrt{2}) - \left[\sqrt{1+x^2} \right]_0^1 = \ln(1+\sqrt{2}) - \sqrt{2} + 1.$

Local review guidance: square completed correctly under the surd (or linear substitution identified); by parts with the inverse function chosen as u ; exact forms with the constant of integration on indefinite integrals. This is authored classroom guidance.