

Exercise walk-through

7.2

The ellipse and the hyperbola

Pearson Edexcel · International A-Level · Pure Mathematics

7. Integrating hyperbolic and inverse functions

Integrating

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh} \frac{x}{a} + c, \quad \int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh} \frac{x}{a} + c, \quad \int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + c,$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c. \quad \text{Complete the square first: } 4x^2 + 8x + 9 = 4((x+1)^2 + \frac{5}{4}).$$

Inverse functions: parts with u the inverse, e.g.

$$\int \operatorname{arcosh} 3x \, dx = x \operatorname{arcosh} 3x - \frac{1}{3} \sqrt{9x^2 - 1} + c.$$

10. Further matrix algebra

Matrices and eigenvalues

3×3 det by expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$; singular when $\det = 0$. Eigen check row

by row: components of $\mathbf{M}\mathbf{v}$ equal λv_j . For $\begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$ with eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$:

$\lambda = 3$, $a = 2$, $b = -1$, others 1 and -2 . Symmetric matrices: orthogonal $\mathbf{P}$ of normalised eigenvectors, $\mathbf{P}^T \mathbf{A} \mathbf{P}$ diagonal.

Check yourself

Check yourself

Derive a cosh identity from exponentials; solve a mixed equation as a quadratic in e^x ; state and prove an arsinh log form; place foci and directrices of both conics; tangent by parametric derivative and by discriminant; differentiate and integrate hyperbolic and inverse functions; build and iterate a reduction formula; arc length and surface exactly; cross and triple products; plane forms and skew distance; 3×3 inverse; eigenvalues, eigenvectors, diagonalisation.

Exercises 2.1·2.3

2.1 For the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, find the eccentricity, the coordinates of the foci, and the equations of the directrices. [3]

2.2 For the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, find the eccentricity, the foci and the directrices. [3]

2.3 Find the equations of the tangents to $\frac{x^2}{4} + \frac{y^2}{3} = 1$ with gradient $\frac{1}{2}$. [4]

Exercises 2.1·2.3

2.1 For the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, find the eccentricity, the coordinates of the foci, and the equations of the directrices. [3]

2.1 $9 = 25(1 - e^2)$ gives $e = \frac{4}{5}$. Foci $(\pm ae, 0) = (\pm 4, 0)$; directrices $x = \pm \frac{a}{e} = \pm \frac{25}{4}$.

2.2 For the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, find the eccentricity, the foci and the directrices. [3]

2.2 $16 = 9(e^2 - 1)$ gives $e^2 = \frac{25}{9}$, $e = \frac{5}{3}$. Foci $(\pm 5, 0)$; directrices $x = \pm \frac{9}{5}$.

2.3 Find the equations of the tangents to $\frac{x^2}{4} + \frac{y^2}{3} = 1$ with gradient $\frac{1}{2}$. [4]

2.3 With $m = \frac{1}{2}$: $c^2 = a^2m^2 + b^2 = 4 \cdot \frac{1}{4} + 3 = 4$, so $c = \pm 2$ and the tangents are $y = \frac{1}{2}x \pm 2$. (Substituting either into the ellipse gives a quadratic with zero discriminant.)

Exercises 2.4·3.1

2.4 Show that the point $(4 \sec \theta, 3 \tan \theta)$ lies on $\frac{x^2}{16} - \frac{y^2}{9} = 1$ for all values of θ . [3]

3.1 (*Edexcel IAL WFM03 June 2025 Q9a–c, condensed.*) The hyperbola H has equation $\frac{x^2}{64} - \frac{y^2}{49} = 1$ and eccentricity e . (a) Show that $e = \frac{\sqrt{113}}{8}$. (b) Use calculus to show that the tangent to H at $P(8 \sec \theta, 7 \tan \theta)$ has equation $\frac{x}{8} \sec \theta - \frac{y}{7} \tan \theta = 1$. (c) Write down the coordinates of the point Q where this tangent meets the y -axis. [7]

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2.4 Show that the point $(4 \sec \theta, 3 \tan \theta)$ lies on $\frac{x^2}{16} - \frac{y^2}{9} = 1$ for all values of θ . [3]

$$\mathbf{2.4} \quad \frac{(4 \sec \theta)^2}{16} - \frac{(3 \tan \theta)^2}{9} = \sec^2 \theta - \tan^2 \theta = 1 \text{ for all } \theta \checkmark.$$

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3.1 (a) $a^2 = 64$, $b^2 = 49$: $49 = 64(e^2 - 1)$, so $e^2 = \frac{113}{64}$ and $e = \frac{\sqrt{113}}{8}$. (b) $x = 8 \sec \theta$, $y = 7 \tan \theta$: $\frac{dy}{dx} = \frac{7 \sec^2 \theta}{8 \sec \theta \tan \theta}$. The tangent: $y - 7 \tan \theta = \frac{7 \sec \theta}{8 \tan \theta}(x - 8 \sec \theta)$; multiplying through by $\frac{8 \tan \theta}{7}$ and collecting, $\frac{x}{8} \sec \theta - \frac{y}{7} \tan \theta = \sec^2 \theta - \tan^2 \theta = 1$. (c) At $x = 0$: $-\frac{y}{7} \tan \theta = 1$, so $Q = (0, -7 \cot \theta)$.

Exercise 3.2

3.2 (*Original.*) The hyperbola K has equation $\frac{x^2}{49} - \frac{y^2}{32} = 1$. (a) Find its eccentricity and the coordinates of its foci. (b) Verify that the point $(7 \sec \theta, 4\sqrt{2} \tan \theta)$ lies on K for all θ . [5]

Exercise 3.2

3.2 (*Original.*) The hyperbola K has equation $\frac{x^2}{49} - \frac{y^2}{32} = 1$. (a) Find its eccentricity and the coordinates of its foci. (b) Verify that the point $(7 \sec \theta, 4\sqrt{2} \tan \theta)$ lies on K for all θ . [5]

3.2 (a) $a^2 = 49$, $b^2 = 32$: $32 = 49(e^2 - 1)$, so $e^2 = \frac{81}{49}$, $e = \frac{9}{7}$; foci $(\pm ae, 0) = (\pm 9, 0)$. (b) $\frac{(7 \sec \theta)^2}{49} = \sec^2 \theta$ and $\frac{(4\sqrt{2} \tan \theta)^2}{32} = \frac{32 \tan^2 \theta}{32} = \tan^2 \theta$, so the left side is $\sec^2 \theta - \tan^2 \theta = 1$ ✓.

Local review guidance: correct b^2 – e relation for the curve type (sign!); foci and directrices placed with ae and a/e ; tangents from $\frac{dy}{dx}$ at the parametric point, or the $c^2 = a^2 m^2 \pm b^2$ condition for given gradients. This is authored classroom guidance.