Further Pure Mathematics 3

Edexcel IAL FP3 (WFM03/01); the final pure unit after P1–P4 + FP1–FP2

Further Pure Mathematics 3: hyperbolic functions, conics, vectors and matrices

- use **hyperbolic functions** and their inverses, and solve equations with them;
- work with the **ellipse** and the **hyperbola**, including their foci and directrices;
- differentiate and integrate hyperbolic and inverse functions, use reduction formulae, and find arc lengths and surface areas;

Further Pure Mathematics 3: hyperbolic functions, conics, vectors and matrices (2)

- use vector products to work with lines and planes in three dimensions;
- find 3×3 determinants, inverses, eigenvalues and eigenvectors, and **diagonalise** symmetric matrices.

Hyperbolic functions

- The graph of $y = \cosh x$ is a **catenary** . It is symmetrical about the y-axis, with a minimum point at $(0, 1)$.
- The graph of $y = \sinh x$ passes through the origin and has rotational symmetry about it.
- The graph of $y = \tanh x$ lies between the **asymptote** lines $y = 1$ and $y = -1$.

$$\cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right) \qquad \sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right) \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

Inverse hyperbolic functions

- $\sinh x$ and $\tanh x$ are one-to-one, so they have inverses.
- $\cosh x$ is one-to-one only for $x \geq 0$, so its inverse uses that part.
- The inverse functions can be written as logarithms:
- You must be able to prove these.

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right),\ x \geq 1 \qquad \operatorname{artanh} x = \tfrac{1}{2}\ln\frac{1}{1}$$

The ellipse and the hyperbola

- For a point P on either curve, the distance from P to a **focus** equals e times the distance from P to the corresponding **directrix** : $PS = e \times PN$.
- The **ellipse** is a closed curve; the **hyperbola** has two branches.
- The number e is the **eccentricity** .
- **Worked example 4.** Find the **eccentricity** and foci of the **hyperbola** $\frac{x^2}{64} - \frac{y^2}{49} = 1$.

Tangents, normals and loci

- at a point given by parameters, differentiate: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$;
- for a line $y = mx + c$, substitute into the equation of the curve and make the discriminant zero. This gives $c^2 = a^2m^2 + b^2$ for the **ellipse** and $c^2 = a^2m^2 - b^2$ for the **hyperbola**.

Tangents, normals and loci (2)

- $\frac{dy}{dx} = \frac{7\sec^2\theta}{8\sec\theta\tan\theta} = \frac{7\sec\theta}{8\tan\theta}$, and the tangent simplifies to $\frac{x\sec\theta}{8} - \frac{y\tan\theta}{7} = \sec^2\theta - \tan^2\theta = 1$.
- The tangent meets the y-axis at $Q(0, -7\cot\theta)$. The normal, $8x\cos\theta + 7y\cot\theta = 113$, meets it at $R\left(0, \frac{113}{7}\tan\theta\right)$.

Tangents, normals and loci (3)

- A circle with diameter QR, where $Q = (0, q)$ and $R = (0, r)$, crosses the x-axis where $x^2 = -qr$. Here $qr = -7\cot\theta \times \frac{113}{7}\tan\theta = -113$, so $x = \pm\sqrt{113}$ for every value of θ.
- These are the foci of H, so the circle always passes through both foci.

Differentiating hyperbolic and inverse functions

- You can prove the last three by writing, for example, $y = \operatorname{arsinh} x$ as $x = \sinh y$ and differentiating implicitly.
- **Worked example 6.** Find the stationary point of $y = \operatorname{arsinh} x + \operatorname{arsinh} \frac{1}{x}$, for $x > 0$.
- (Because $x > 0$, $\sqrt{x^2} = x$.) $\frac{dy}{dx} = 0$ when $x = 1$, and then $y = 2 \operatorname{arsinh} 1 = 2 \ln \left(1 + \sqrt{2}\right)$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + \frac{-1/x^2}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+x^2}} - \frac{1}{x\sqrt{1+x^2}} = \frac{x-1}{x\sqrt{1+x^2}}$$

Integrating hyperbolic and inverse functions

- For a square root of a quadratic in the denominator, complete the square first.
- To integrate an inverse function, use integration by parts with $u =$ the inverse function and $\dfrac{dv}{dx} = 1$.
- **Worked example 7.** Find $\displaystyle\int \frac{1}{\sqrt{4x^2+8x+9}}\,dx$.
- Since $4x^2+8x+9 = 4\left((x+1)^2+\frac{5}{4}\right)$,

$$\int \frac{dx}{\sqrt{4x^2+8x+9}} = \frac{1}{2}\int \frac{dx}{\sqrt{(x+1)^2+\frac{5}{4}}} = \frac{1}{2}\operatorname{arsinh}\frac{2(x+1)}{\sqrt{5}} + c$$

Substitutions, reduction formulae and surfaces of revolution

- With $u = x^n$ and $v = \frac{2}{3}\sqrt{3x-2}$:
- The bracket is $\dfrac{8\times 6^n - 2}{3}$. Write $\sqrt{3x-2} = \dfrac{3x-2}{\sqrt{3x-2}}$, so the integral is $3I_n - 2I_{n-1}$.
- $I_n = \dfrac{8\times 6^n - 2}{3} - \dfrac{2n}{3}(3I_n - 2I_{n-1})$. Multiply by 3 and collect the I_n terms: $(3+6n)I_n = 4nI_{n-1} + 8\times 6^n - 2$.

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\frac{x}{a} + c \qquad \int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh}\frac{x}{a} + c \quad (x > a)$$

Substitutions, reduction formulae and surfaces of revolution (2)

- $I_0 = \left[\frac{2}{3}\sqrt{3x-2}\right]_1^6 = 2$. Then $9I_1 = 8 + 48 - 2$, so $I_1 = 6$; $15I_2 = 48 + 288 - 2 = 334$; and $21I_3 = 12 \times \frac{334}{15} + 1728 - 2$, so $I_3 = \frac{3322}{35}$.
- $\dfrac{dy}{dx} = -2\sin 2x$, so $S = 2\pi \displaystyle\int_0^{\pi/4} \cos 2x\sqrt{1 + 4\sin^2 2x}\,dx$.
- $\cosh\theta\, d\theta = 4\cos 2x\, dx$, and $\sqrt{1 + 4\sin^2 2x} = \cosh\theta$. The limits become 0 and $\operatorname{arsinh} 2$.

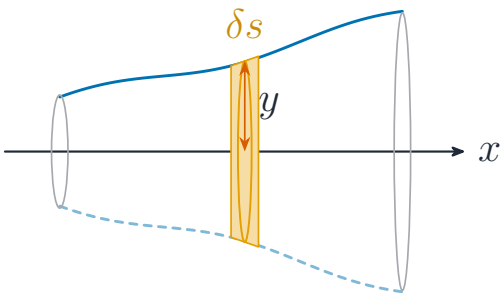
$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + c \qquad \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

Substitutions, reduction formulae and surfaces of revolution (3)

- $S = \frac{\pi}{2}\int_0^{\operatorname{arsinh} 2} \cosh^2\theta \, d\theta = \frac{\pi}{4}\Big[\sinh\theta\cosh\theta + \theta\Big]_0^{\operatorname{arsinh} 2}$.
- At the upper limit, $\sinh\theta = 2$ and $\cosh\theta = \sqrt{5}$, so $S = \frac{\pi}{4}\left(2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right)$.

$$I_n = \left[\tfrac{2}{3}x^n\sqrt{3x-2}\right]_1^6 - \tfrac{2n}{3}\int_1^6 x^{n-1}\sqrt{3x-2}\,dx$$

Figure: Substitutions, reduction formulae and surfaces of revolution



band area $\approx 2\pi y\,\delta s$

$$S = 2\pi \int |y| \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- A **surface of revolution** is made of thin bands, each with area about $2\pi y \delta s$.
- Read the axes and units first.

Vectors: products, lines and planes

- A line: $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, or $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.
- A plane: $\mathbf{r} \cdot \mathbf{n} = p$, or $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$, where $\mathbf{n} = \mathbf{b} \times \mathbf{c}$ is a normal.
- The distance from the point with position vector $\mathbf{x}$ to the plane $\mathbf{r} \cdot \mathbf{n} = p$ is $\dfrac{|\mathbf{x} \cdot \mathbf{n} - p|}{|\mathbf{n}|}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

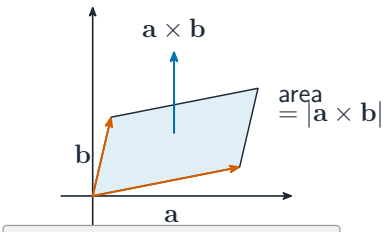
Vectors: products, lines and planes (2)

- **Skew lines** are not parallel and do not meet. To find the shortest distance between them, find the common normal $\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2$, and project the vector between a point on each line onto it.
- Normal to Π_1: $(7, 5, -6) \times (-3, 1, 2) = (16, 4, 22) = 2(8, 2, 11)$. Using the point $(4, 2, -1)$: $\mathbf{r} \cdot (8, 2, 11) = 32 + 4 - 11 = 25$.

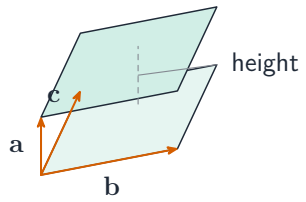
Vectors: products, lines and planes (3)

- The line of intersection has direction $(8, 2, 11) \times (1, -1, 1) = (13, 3, -10)$, and the point $\left(\frac{13}{5}, -\frac{17}{5}, 1\right)$ lies on both planes. So its equation is $\left(\mathbf{r} - \left(\frac{13}{5}, -\frac{17}{5}, 1\right)\right) \times (13, 3, -10) = \mathbf{0}$.
- The line through $A(2, 1, 3)$ and $B(3, 0, 2)$ has direction $(1, -1, -1)$. The common normal is $(13, 3, -10) \times (1, -1, -1) = (-13, 3, -16)$, with length $\sqrt{434}$.
- The vector from $\left(\frac{13}{5}, -\frac{17}{5}, 1\right)$ to A is $\left(-\frac{3}{5}, \frac{22}{5}, 2\right)$. Shortest distance $= \dfrac{\left|\frac{39}{5} + \frac{66}{5} - 32\right|}{\sqrt{434}} = \dfrac{11}{\sqrt{434}}$.

Figure: Vectors: products, lines and planes



$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$
is the parallelogram area



$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$: base area × height

- The **vector product** gives the area of a parallelogram (left); the **scalar triple product** gives the volume of a parallelepiped (right).
- Match each part of the diagram to the text.

Further matrix algebra

- Row 1 of $\mathbf{Mv}$: $1 - 2 + 4 = 3$, which must be $\lambda \times 1$, so $\lambda = 3$.
- Row 2: $3 + 2a + b = 3 \times 2 = 6$. Row 3: $a + 2 + b = 3 \times 1 = 3$. So $2a + b = 3$ and $a + b = 1$, giving $a = 2$, $b = -1$.
- The sum of the eigenvalues equals the trace, $1 + 2 - 1 = 2$, and their product equals $\det \mathbf{M} = 1(-1) + 1(-1) + 4(-1) = -6$. The other two eigenvalues add to -1 and multiply to -2, so they are 1 and -2.

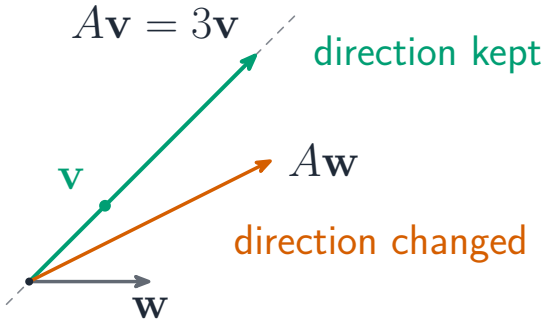
$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj} \mathbf{A}$$

Further matrix algebra (2)

- For $\lambda = 1$, solving $(\mathbf{M} - \mathbf{I})\mathbf{v} = \mathbf{0}$ gives $\mathbf{v} = (-1, 4, 1)$. For $\lambda = -2$, $(\mathbf{M} + 2\mathbf{I})\mathbf{v} = \mathbf{0}$ gives $\mathbf{v} = (-1, 1, 1)$.
- $\det \mathbf{A} = 1(12 - 2) - k(20 + 12) + 2(-5 - 18) = -36 - 32k$, so $\mathbf{A}$ is **singular** when $k = -\frac{9}{8}$.
- Using the transposed cofactors,
- Check: row 1 of $\mathbf{A}$ times column 1 of the adjugate is $10 - 32k - 46 = -36 - 32k$, which equals the determinant. ✓

$$\mathbf{A}^{-1} = \frac{1}{-36 - 32k} \begin{pmatrix} 10 & -4k - 2 & -2k - 6 \\ -32 & -8 & 12 \\ -23 & 6k + 1 & 3 - 5k \end{pmatrix}$$

Figure: Further matrix algebra



- An **eigenvector** keeps its direction under the matrix; other vectors are turned.
- Match each part of the diagram to the text.

Sources for this lesson

- Pearson Edexcel, *International Advanced Level Mathematics, Further Mathematics and Pure Mathematics specification* (2018), unit FP3, and the formula booklet.
- G. W. Leibniz, C. Huygens and J. Bernoulli, solutions of the **catenary** problem, *Acta Eruditorum* (Leipzig, June 1691).
- Pearson Edexcel, WFM03/01 June 2025, questions 1 to 9 (adapted).

Common errors

- Confusing the hyperbolic identities with the trigonometric ones: $\cosh^2 x - \sinh^2 x = 1$, with a minus sign.
- Forgetting to reject the negative root when solving a quadratic in e^x.
- Mixing up $b^2 = a^2(1 - e^2)$ for the **ellipse** and $b^2 = a^2(e^2 - 1)$ for the **hyperbola**.

Common errors (2)

- Forgetting the domain of arcosh and artanh.
- Making a sign error in the middle component of a **vector product**.
- Using the wrong sign pattern for cofactors, or forgetting to transpose them.
- Giving an **eigenvector** as the zero vector.

Key terms

- **Hyperbolic functions**: $\cosh$, $\sinh$ and $\tanh$, defined using e^x and e^{-x}.
- **Eccentricity**: the ratio e in $PS = e \times PN$; $e < 1$ for an **ellipse** and $e > 1$ for a **hyperbola**.
- **Reduction formula**: a formula linking I_n to I_{n-1} or I_{n-2}.

Key terms (2)

- **Vector product**: a vector perpendicular to two given vectors, with length equal to the parallelogram area.
- **Scalar triple product**: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, the volume of a parallelepiped.
- **Eigenvector**: a non-zero vector $\mathbf{v}$ with $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$; λ is its **eigenvalue**.

Check yourself

- 1.
- Solve $\sinh x = 2$, giving your answer as a logarithm. 2.
- Prove that $\cosh^2 x - \sinh^2 x = 1$. 3.
- Find the **eccentricity** and the foci of the **ellipse** $\frac{x^2}{25} + \frac{y^2}{16} = 1$. 4.