Further Pure Mathematics 1

Edexcel IAL FP1 (WFM01/01); compulsory for IAS/IAL Further Mathematics and Pure Mathematics

Further Pure Mathematics 1: complex numbers, matrices and proof

- calculate with complex numbers, and find the complex roots of polynomials with real coefficients;
- use the sum and product of the roots of a quadratic, and solve equations numerically;
- work with the **parabola** and the **rectangular hyperbola**;

Further Pure Mathematics 1: complex numbers, matrices and proof (2)

- use matrices to represent and combine transformations;
- sum series and prove results by mathematical **induction**.

Complex numbers

- $5r = 15$, so $r = 3$ and $z_1 = 3\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$.
- For $z_1 + z_2$ to be real, the imaginary parts must cancel, so $z_2 = a + \frac{3}{2}i$.
- $a^2 + \frac{9}{4} = 25$, so $a = \pm\frac{\sqrt{91}}{2}$, and $z_2 = \pm\frac{\sqrt{91}}{2} + \frac{3}{2}i$.

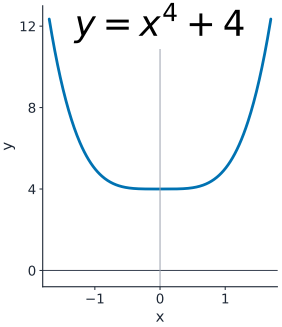
$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*}$$

Complex roots of polynomials

- $3 - 5i$ is also a root. The sum of the pair is 6 and the product is $9 + 25 = 34$, so $z^2 - 6z + 34$ is a factor.
- Write $f(z) = (z^2 - 6z + 34)(Pz^2 + bz + c)$ and compare coefficients: - constant: $34c = 68$, so $c = 2$; - z: $34b - 6c = 192$, so $b = 6$; - z^3: $b - 6P = -36$, so $P = 7$; - z^2: $Q = c - 6b + 34P = 2 - 36 + 238 = 204$.
- The other roots solve $7z^2 + 6z + 2 = 0$: $z = \dfrac{-6 \pm \sqrt{36 - 56}}{14} = \dfrac{-3 \pm i\sqrt{5}}{7}$.

$$(z - z_1)(z - z_1^*) = z^2 - (\text{sum})z + \text{product} = z^2 - 2\,\mathrm{Re}(z_1)\,z + |z_1|^2$$

Figure: Complex roots of polynomials



No x-intercepts

Four complex roots

$1 + i, \quad 1 - i$
$-1 + i, \quad -1 - i$

Sum = 0
Product = 4

- The graph of $y = x^4 + 4$ never meets the x-axis, but the equation has four complex roots, $\pm 1 \pm i$.
- Read the axes and units first.

Roots of quadratic equations

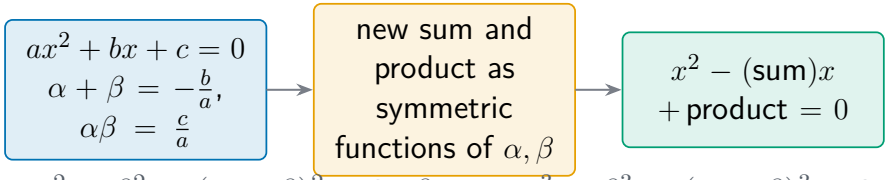
- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta}$

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Roots of quadratic equations (2)

- $\alpha\beta = -\frac{5}{3}$, so $\alpha + \beta = -15$. Since $\alpha + \beta = -\frac{k}{3}$, $k = 45$.
- Sum $= (\alpha^2 + \beta^2) + (\alpha + \beta) = \left(225 + \frac{10}{3}\right) - 15 = \frac{640}{3}$.
- Product $= \alpha^3 + \beta^3 + \alpha\beta + (\alpha\beta)^2$. Here $\alpha^3 + \beta^3 = (-15)^3 - 3\left(-\frac{5}{3}\right)(-15) = -3375 - 75 = -3450$, so the product is $-3450 - \frac{5}{3} + \frac{25}{9} = -\frac{31\,040}{9}$.
- Multiply $x^2 - \frac{640}{3}x - \frac{31\,040}{9} = 0$ by 9: $9x^2 - 1920x - 31\,040 = 0$.

Figure: Roots of quadratic equations



$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \qquad \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

no individual root is ever needed

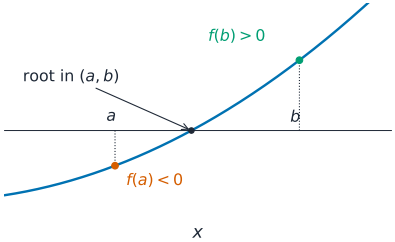
- Forming an equation with new roots from the sum and product of the old ones.
- Match each part of the diagram to the text.

Numerical solution of equations

- **Interval bisection** : find the sign of f at the midpoint of the interval, and keep the half that still has a change of sign.
- **Linear interpolation** : join $(a, f(a))$ and $(b, f(b))$ with a straight line and use the point where it crosses the x-axis:
- The **Newton–Raphson method** - (in the formula booklet) uses the tangent: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. It fails if $f'(x_n) = 0$, and it may not converge from a poor starting value.

$$x \approx a + \frac{(b-a)\,|f(a)|}{|f(a)| + |f(b)|}$$

Figure: Numerical solution of equations



if f continuous on $[a, b]$ and $f(a)f(b) < 0$, then a root lies in (a, b)

- A change of sign of a continuous function on $[a, b]$ shows that a root lies between a and b.
- Read the axes and units first.

Parabola and rectangular hyperbola

- $y^2 = 4ax$ gives $2y\dfrac{dy}{dx} = 4a$, so $\dfrac{dy}{dx} = \dfrac{2a}{y} = \dfrac{1}{t}$ at the point with parameter t;
- $xy = c^2$ gives $y = \dfrac{c^2}{x}$, so $\dfrac{dy}{dx} = -\dfrac{c^2}{x^2} = -\dfrac{1}{t^2}$ at $(ct, \frac{c}{t})$.

$$y - \frac{9}{t} = t^2(x - 9t) \quad \Longrightarrow \quad ty = t^3x + 9(1 - t^4)$$

Matrix algebra

- Matrices of the same size are added entry by entry.
- To multiply, combine rows of the first matrix with columns of the second.
- In general $\mathbf{AB} \neq \mathbf{BA}$.
- For a 2×2 matrix, $\det \mathbf{M} = ad - bc$.

$$\mathbf{M}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$$

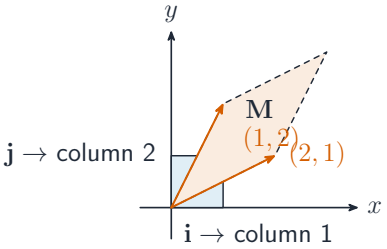
Transformations using matrices

- $\mathbf{AB}$ means: apply $\mathbf{B}$ first, then $\mathbf{A}$.
- $|\det \mathbf{M}|$ is the **area scale factor** . A negative determinant means the transformation includes a reflection.
- $\mathbf{P}$ maps $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$: a rotation of $90°$ clockwise about O. $\mathbf{Q}$ is an enlargement with scale factor 5, centre O.

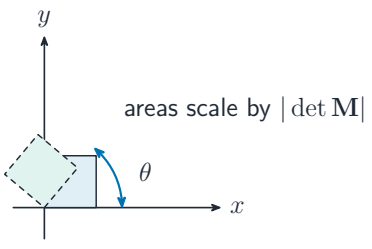
Transformations using matrices (2)

- Reflecting $(4, 3)$ in $y = -x$ gives $A(-3, -4)$.
- $\mathbf{P}$ followed by $\mathbf{Q}$ maps B to A. Then $B = \mathbf{P}^{-1}\mathbf{Q}^{-1}A$. $\mathbf{Q}^{-1}A = (-0.6, -0.8)$, and $\mathbf{P}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so $B = (0.8, -0.6)$.

Figure: Transformations using matrices



$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: columns are the images of $\mathbf{i}, \mathbf{j}$



rotation by θ: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

- The columns of the matrix are the images of the unit vectors; the determinant is the **area scale factor**.
- Match each part of the diagram to the text.

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Series

- You can find the sum of any polynomial series using the standard results $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and, from the formula booklet, $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ and $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.
- For a sum to $2n$, replace n by $2n$ in the formulae.
- **Worked example 13.** Show that $\sum_{r=1}^{2n}(2r^2-1) = \frac{4}{3}n(n+1)(4n-1)$.
- and $4n^2+3n-1 = (n+1)(4n-1)$, which gives the result.

$$2\sum_{r=1}^{2n} r^2 - 2n = 2\cdot\frac{2n(2n+1)(4n+1)}{6} - 2n = \frac{2n(8n^2+6n+1)-6n}{3} = \frac{4n(4n^2+3n-1)}{3}$$

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Proof by mathematical induction

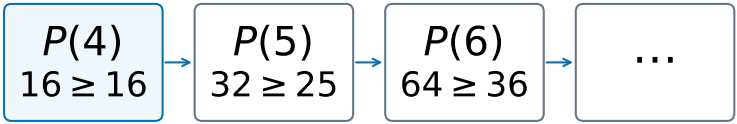
- $n = 1$: the left side is $1 \times 1^2 = 1$ and the right side is $\frac{1}{4} \times 1 \times 2 \times 2 = 1$. True.
- Assume it is true for $n = k$. Add the $(k+1)$th term, $(k+1)(3k+1)^2$:
- The result for $n = k+1$ is
  $\frac{1}{4}(k+1)^2(k+2)(9k+2) = \frac{1}{4}(k+1)\left[(k+1)(k+2)(9k+2)\right]$, and
  $(k+1)(k+2)(9k+2) = 9k^3 + 29k^2 + 24k + 4$. The two are equal.
- So it is true for $n = k+1$, and by **induction** for all positive integers n.

$$\frac{1}{4}k^2(k+1)(9k-7) + (k+1)(3k+1)^2 = \frac{1}{4}(k+1)\left[9k^3 + 29k^2 + 24k + 4\right]$$

Figure: Proof by mathematical induction

Claim: $2^n \geq n^2$ for integers $n \geq 4$

For every $k \geq 4$: $P(k) \Rightarrow P(k+1)$



Base case Reached by the proved implication

Checked values alone do not justify the arrows.

- An **induction** proof can start at any first value, here = 4$; the general step then links each case to the next.
- Match each part of the diagram to the text.

Sources for this lesson

- Pearson Edexcel, *International Advanced Level Mathematics, Further Mathematics and Pure Mathematics specification* (2018), unit FP1, and the formula booklet.
- G. Cardano, *Ars Magna* (Nuremberg, 1545), chapter 37; R. Bombelli, *L'Algebra* (Bologna, 1572).
- Pearson Edexcel, WFM01/01 June 2025, questions 1 to 4 and 6 to 9 (adapted).

Common errors

- Writing $i^2 = 1$, or forgetting to multiply the denominator by its **conjugate** as well.
- Giving the **argument** in the wrong quadrant; draw the **Argand diagram** first.
- Forgetting the conjugate root of a polynomial with real coefficients.

Common errors (2)

- Using $\alpha + \beta = \dfrac{b}{a}$ instead of $-\dfrac{b}{a}$.
- Giving a root from a calculator's solver without showing the values of f.
- Multiplying matrices in the wrong order: $\mathbf{AB}$ means $\mathbf{B}$ first.
- Leaving out the concluding sentence of an **induction** proof.

Key terms

- **Complex conjugate**: $z^* = a - bi$ for $z = a + bi$.
- **Modulus**: the distance of z from the origin on an **Argand diagram**.
- **Argument**: the angle of z measured from the positive real axis.

Key terms (2)

- **Singular matrix**: a matrix with determinant 0, which has no inverse.
- **Area scale factor**: $|\det \mathbf{M}|$, the factor by which a transformation multiplies areas.
- **Mathematical induction**: a proof that a statement for $n = 1$, and the step from $n = k$ to $n = k + 1$, together prove it for all positive integers.

Check yourself

- 1.
- Find $(2+3i)(4-i)$ and $\dfrac{5}{2+i}$. 2.
- Find the **modulus** and **argument** of $-1+i$. 3.
- $2-i$ is a root of $z^3-3z^2+z+5=0$.