2 · Pure Mathematics 2

Pearson Edexcel · International A-Level · Pure Mathematics

Pure Mathematics 2: building on the P1 toolkit

Pure Mathematics 2 (P2) is the second compulsory unit of International AS Mathematics and International AS Pure Mathematics, and of both International A Level routes. This reference covers every P2 topic in the Edexcel IAL specification (Issue 3, April 2019): proof, the factor and remainder theorems, circle geometry, sequences and series, exponentials and logarithms, trigonometric identities and equations, stationary points, and integration including areas and the trapezium rule.

- The P2 paper (WMA12/01) lasts **1 hour 30 minutes** with **75 marks**. A knowledge of P1 and its formulae is assumed and **may be tested** on the P2 paper.
- The same calculator and yellow *Mathematical Formulae and Statistical Tables* rules apply as for P1. The formulae below do **not** appear in the booklet – memorise them.
- "Show all stages of your working" questions reject calculator-only answers; give inexact answers to 3 significant figures unless told otherwise.

Formulae you must know for P2 (from the specification's assessment information):

Formula	Use
$\log_a x + \log_a y \equiv \log_a(xy)$, $\log_a x - \log_a y \equiv \log_a \dfrac{x}{y}$, $k\log_a x \equiv \log_a(x^k)$	laws of logarithms
$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$, $\sin^2\theta + \cos^2\theta \equiv 1$	trigonometric identities
area under a curve $= \displaystyle\int_a^b y\,dx \quad (y \geq 0)$	definite integral as an area

Proof

A **proof** proceeds from given assumptions through logical steps to a conclusion. P2 uses three methods:

- **Proof by exhaustion** : split the possibilities into cases and check every case.
- **Disproof by counterexample** : one example that breaks the claim disproves it.

- Algebraic proof: start from what is given, manipulate identities (not equations to solve), and end with the target statement.

Worked example – exhaustion (adapted from Edexcel IAL WMA12 June 2025 Q10). Prove that for all positive integers m that are not multiples of 3, the value of $m^2 + 3m + 2$ is always a multiple of 3.

- Factorise first: $m^2 + 3m + 2 = (m+1)(m+2)$.
- A positive integer not divisible by 3 is either $m = 3q+1$ or $m = 3q+2$ – those are all the cases, so exhausting them proves the claim.
- Case $m = 3q+1$: $(m+1)(m+2) = (3q+2)(3q+3) = 3(3q+2)(q+1)$ – a multiple of 3.
- Case $m = 3q+2$: $(m+1)(m+2) = (3q+3)(3q+4) = 3(q+1)(3q+4)$ – a multiple of 3.
- Both cases give a multiple of 3, so $3 \mid m^2 + 3m + 2$ whenever $3 \nmid m$. ■
- Check: $m = 1$: $6 = 3 \times 2$ ✓; $m = 2$: $12 = 3 \times 4$ ✓; $m = 4$: 30 ✓; $m = 5$: 42 ✓.

Worked example – counterexample. Disprove: "$n^2 - n + 1$ is prime for all positive integers n."

- One counterexample is enough. Try $n = 5$: $25 - 5 + 1 = 21 = 3 \times 7$, which is not prime.
- The claim is false. Check $n = 1, 2, 3, 4$ give $1, 3, 7, 13$ – the pattern holds at first, which is exactly why a counterexample, not more examples, settles the question.

Algebra: the factor and remainder theorems

For a polynomial $f(x)$:

- **Factor theorem** : $(x - b)$ is a factor of $f(x)$ exactly when $f(b) = 0$. For divisors of the form $(ax - b)$: if $f\left(\frac{b}{a}\right) = 0$ then $(ax - b)$ is a factor.
- **Remainder theorem** : dividing $f(x)$ by $(ax - b)$ leaves remainder $f\left(\frac{b}{a}\right)$.
- **Algebraic division** by $(ax \pm b)$ gives a **quotient** and a **remainder**; the degree drops by one: $f(x) = (ax - b) \times (\text{quotient}) + \text{remainder}$.

Worked example – both theorems at once (adapted from Edexcel IAL WMA12 June 2025 Q4). $f(x) = ax^3 + bx^2 + 5x - 3$, where $(x + 3)$ is a factor of $f(x)$ and dividing $f(x)$ by $(2x - 1)$ leaves remainder 7.

(a) Show that $b = 3a + 2$.

- Factor theorem: $f(-3) = 0$:

$$a(-27) + b(9) + 5(-3) - 3 = 0 \implies -27a + 9b - 18 = 0 \implies b = 3a + 2$$

(b) Find a and b.

- Remainder theorem with $(2x-1)$: $f\left(\frac{1}{2}\right)=7$:

$$\frac{a}{8}+\frac{b}{4}+\frac{5}{2}-3=7 \implies \frac{a}{8}+\frac{b}{4}=7.5 \implies a+2b=60$$

- Substitute $b=3a+2$: $a+6a+4=60$, so $a=8$ and $b=26$.
- Check: $f(x)=8x^3+26x^2+5x-3$; $f(-3)=-216+234-15-3=0$ ✓ and $f\left(\frac{1}{2}\right)=1+6.5+2.5-3=7$ ✓.

(c) Quotient and remainder when $f(x)$ is divided by $(x-2)$.

- Divide $8x^3+26x^2+5x-3$ by $(x-2)$: quotient $8x^2+42x+89$, remainder 175.
- Check by rebuilding: $(x-2)(8x^2+42x+89)+175=8x^3+42x^2+89x-16x^2-84x-178+175=8x^3+26x^2+5x-3$ ✓. The remainder also equals $f(2)=64+104+10-3=175$ ✓.

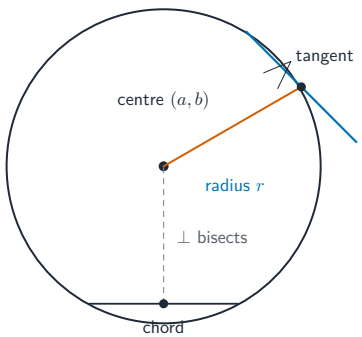
Coordinate geometry of the circle

A circle centre (a,b) radius r has equation

$$(x-a)^2+(y-b)^2=r^2$$

Given the expanded form $x^2+y^2+fx+gy+c=0$, complete the square to recover centre $\left(-\frac{f}{2},-\frac{g}{2}\right)$ and $r^2=\frac{f^2+g^2}{4}-c$. Three circle properties are examined:

- the angle in a semicircle is a right angle;
- the perpendicular from the centre to a **chord** bisects the chord;
- a **tangent** is perpendicular to the radius at the point of contact.



Tangent ⊥ radius at the contact point; the perpendicular from the centre to a chord **bisects** it; the angle in a semicircle is 90.

Radius ⊥ tangent; centre-to-chord perpendicular bisects the chord.

Worked example – diameter endpoints (adapted from Edexcel IAL WMA12 June 2025 Q2). The line joining $(-2,5)$ and $(4,15)$ is a diameter of circle C.

(a) Equation of C.

- The centre is the midpoint: $\left(\frac{-2+4}{2},\frac{5+15}{2}\right)=(1,10)$.
- The diameter squared: $d^2=(4-(-2))^2+(15-5)^2=36+100=136$, so $r^2=136\div 4=34$.

$$(x-1)^2+(y-10)^2=34$$

- Check: both endpoints satisfy the equation: $(-3)^2+(-5)^2=34$ ✓ and $3^2+5^2=34$ ✓.

(b) Exact coordinates of the point on C nearest the x-axis.

- The nearest point lies on the vertical radius: same x as the centre, r below it:

$$(1,\ 10-\sqrt{34})$$

- Check: $(1-1)^2+(10-\sqrt{34}-10)^2=34$ ✓, and no smaller y is possible since the centre sits $10>\sqrt{34}$ above the axis.

Sequences and series

A **sequence** may be given by a formula for the nth term or by a relation $x_{n+1}=f(x_n)$. Describe it as **increasing**, **decreasing** or **periodic** (from its terms, not its formula).

Arithmetic sequences: constant difference d.

$$a_n=a+(n-1)d \qquad S_n=\frac{n}{2}\big(2a+(n-1)d\big) \qquad \sum_{r=1}^{n} r=\frac{n(n+1)}{2}$$

The proof of the sum formula (write S_n forwards and backwards, add) should be known, as should Σ notation.

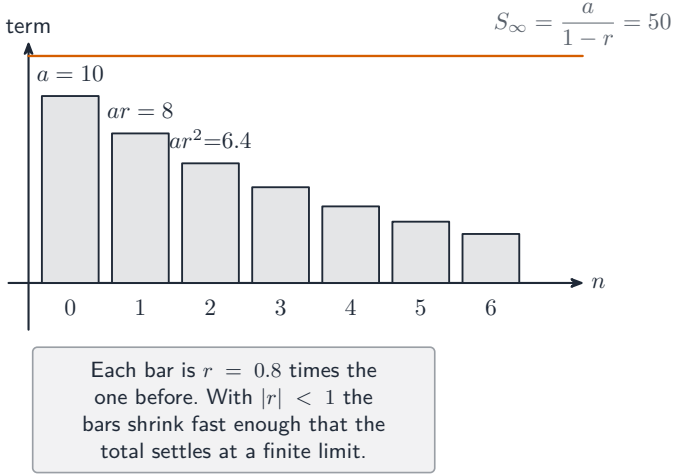
Geometric sequences: constant ratio r.

$$a_n=ar^{n-1} \qquad S_n=\frac{a(1-r^n)}{1-r} \qquad S_\infty=\frac{a}{1-r}\ (|r|<1)$$

The sum formula's proof (telescoping S_n-rS_n) should be known. Use **logarithms** to solve for n when a term value is given.

Binomial expansion for a positive integer n:

$$(a+bx)^n=a^n+\binom{n}{1}a^{n-1}(bx)+\binom{n}{2}a^{n-2}(bx)^2+\cdots+(bx)^n, \qquad \binom{n}{r}=\frac{n!}{r!\,(n-r)!}$$



|r| < 1 means the terms shrink; the sum settles at S∞.

Worked example – geometric series with logs (adapted from Edexcel IAL WMA12 June 2025 Q7). A geometric series begins $10 + 8 + 6.4 + \cdots$

(a) Sum to infinity.

- $a = 10$, $r = 8/10 = 0.8$, and $|r| < 1$:

$$S_\infty = \frac{a}{1-r} = \frac{10}{1-0.8} = 50$$

(b) The kth term is less than 0.0005 – find the smallest k, by algebra.

- The kth term is $10(0.8)^{k-1}$:

$$10(0.8)^{k-1} < 0.0005 \implies (0.8)^{k-1} < 0.00005$$

- Take logs (the log of a number below 1 is negative – keep the inequality direction when dividing by it):

$$(k-1)\log 0.8 < \log 0.00005 \implies k - 1 > \frac{\log 0.00005}{\log 0.8} = 44.38\ldots$$

- So $k > 45.38$ and the smallest integer is $k = 46$.
- Check: $10(0.8)^{45} = 0.00045\ldots < 0.0005$ ✓ but $10(0.8)^{44} = 0.00056 > 0.0005$ ✓.

Worked example – maximum arithmetic sum (adapted from the same question). An arithmetic series begins $850 + 843 + 836 + \cdots$ $(d = -7)$. Its greatest partial sum:

- Terms stay positive until $u_n = 850 - 7(n-1) > 0$, so $n \le 122.4$; the 122nd term is $850 - 7(121) = 3 > 0$ and the 123rd is $-4 < 0$.

$$S_{122} = \frac{122}{2}\big(2(850) + 121(-7)\big) = 61(1700 - 847) = 61 \times 853 = 52\,033$$

- Check: adding a negative 123rd term would reduce the sum, so S_{122} is the maximum ✓.

Exponentials and logarithms

$y = a^x$ $(a > 0,\ a \neq 1)$ passes through $(0, 1)$ and grows without bound for $a > 1$. The **logarithm** $\log_a x$ inverts it: $\log_a x = y$ means $a^y = x$. Know:

$$\log_a x + \log_a y \equiv \log_a(xy) \qquad \log_a x - \log_a y \equiv \log_a \frac{x}{y} \qquad k\log_a x \equiv \log_a(x^k) \qquad \log_a a = 1$$

and $\log_a \frac{1}{x} = -\log_a x$. Solve $a^x = b$ with $x = \log_a b$ (a change of base may be used). Always check candidates against the **domain** : logs need positive arguments.

Worked example – a log equation (adapted from Edexcel IAL WMA12 June 2025 Q6). Solve $2\log_2(4-x) = 3 + \log_2\left(\frac{x+11}{2}\right)$.

- Bring everything to one log per side:

$$\log_2(4-x)^2 = \log_2 8 + \log_2\left(\frac{x+11}{2}\right) = \log_2\big(4(x+11)\big)$$

- Equal logs give equal arguments:

$$(4-x)^2 = 4(x+11) \implies x^2 - 8x + 16 = 4x + 44 \implies x^2 - 12x - 28 = 0$$

$$(x-14)(x+2) = 0 \implies x = 14 \text{ or } x = -2$$

- Domain check: $4 - x > 0$ needs $x < 4$, so $x = 14$ is rejected. $x = -2$.
- Check: LHS $= 2\log_2 6$; RHS $= 3 + \log_2(4.5) = \log_2(8 \times 4.5) = \log_2 36 = 2\log_2 6$ ✓.

Worked example – exact intersection in $\log_3$ form (same question). The curves $y = 3^{2x+1}$ and $y = 6 \times 3^x$ meet at P; write P as $(\log_3 a, b)$.

- Set equal and substitute $u = 3^x$:

$$3 \cdot 3^{2x} = 6 \cdot 3^x \implies 3u^2 = 6u \implies u = 2 \implies 3^x = 2 \implies x = \log_3 2$$

$$y = 6 \times 3^x = 6 \times 2 = 12$$

- $P = (\log_3 2,\ 12)$, so $a = 2$, $b = 12$. Check: $3^{2\log_3 2+1} = 3 \cdot 3^{2\log_3 2} = 3 \cdot 4 = 12$ ✓.

Trigonometric identities and equations

Two identities convert between sine, cosine and tangent:

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta \equiv 1$$

Solve **trigonometric equations** in a given interval by reducing to one function (often a hidden quadratic), then reading the second solution from the graph's symmetry. State the interval, work in its units (degrees or radians), and reject values outside.

Worked example – radians and exact fractions (adapted from Edexcel IAL WMA12 June 2025 Q8). Solve $3\tan\left(2x + \frac{\pi}{5}\right) = \sqrt{3}$ for $0 \le x < \pi$.

- Divide by 3: $\tan\left(2x + \frac{\pi}{5}\right) = \frac{1}{\sqrt{3}}$, and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$:

$$2x + \frac{\pi}{5} = \frac{\pi}{6} + k\pi \implies 2x = -\frac{\pi}{30} + k\pi \implies x = -\frac{\pi}{60} + \frac{k\pi}{2}$$

- In $0 \le x < \pi$: $x = \frac{29\pi}{60}$ (k = 1) and $x = \frac{59\pi}{60}$ (k = 2).
- Check: at $x = \frac{29\pi}{60}$: $2x + \frac{\pi}{5} = \frac{7\pi}{6}$ and $\tan\frac{7\pi}{6} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ ✓.

Worked example – hidden quadratic in cosine (same question). Solve $5\sin\theta\tan\theta = \cos\theta + 4$ for $0 \le \theta < 360°$.

- Replace $\tan\theta = \sin\theta/\cos\theta$ and multiply through by $\cos\theta$:

$$\frac{5\sin^2\theta}{\cos\theta} = \cos\theta + 4 \implies 5\sin^2\theta = \cos^2\theta + 4\cos\theta$$

- Use $\sin^2\theta = 1 - \cos^2\theta$:

$$5(1 - \cos^2\theta) = \cos^2\theta + 4\cos\theta \implies 6\cos^2\theta + 4\cos\theta - 5 = 0$$

- Quadratic formula in $\cos\theta$:

$$\cos\theta = \frac{-4 \pm \sqrt{16 + 120}}{12} = \frac{-2 \pm \sqrt{34}}{6}$$

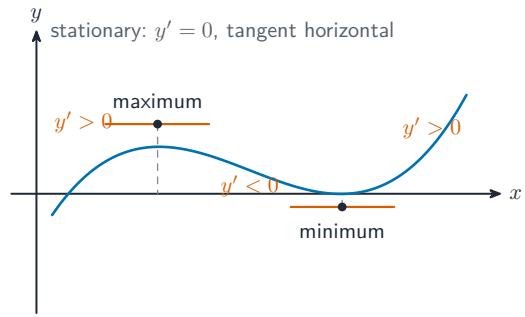
- The $-$ root is below -1 (reject); $\cos\theta = 0.6385$ gives $\theta = 50.3°$ and, from cosine's symmetry, $\theta = 360° - 50.3° = 309.7°$.
- Check: $\cos 50.3° = 0.638$ ✓; both angles lie in the interval ✓.

Differentiation: stationary points

Stationary points have $\frac{dy}{dx} = 0$. Classify with the sign of the derivative:

- derivative changes $+$ to $-$: **maximum** ;
- derivative changes $-$ to $+$: **minimum** ;
- the second derivative helps: $f''(x) > 0$ at the point confirms a minimum, $f''(x) < 0$ a maximum.

A function is **increasing** where $f'(x) > 0$ and **decreasing** where $f'(x) < 0$. Practical problems – least material, least cost – translate into minimising an expression in one variable.



Stationary means the tangent is horizontal; the sign change classifies it.

Worked example – a modelling minimum (adapted from Edexcel IAL WMA12 June 2025 Q9). An open trough has right-triangle ends with legs $3x$ and $2x$ (metres), rectangular sides $3x \times L$ and $2x \times L$, and capacity 12 m^3. The metal area is S m^2.

(a) Show $S = 6x^2 + \frac{20}{x}$.

- Volume: cross-section $\times$ length $= \frac{1}{2}(3x)(2x) \times L = 3x^2L = 12$, so $L = \frac{4}{x^2}$.
- Surface: two triangles plus the two rectangles:

$$S = 2 \times \tfrac{1}{2}(3x)(2x) + 3xL + 2xL = 6x^2 + 5x \cdot \frac{4}{x^2} = 6x^2 + \frac{20}{x}$$

(b) Minimum S by calculus.

$$\frac{dS}{dx} = 12x - \frac{20}{x^2} = 0 \implies 12x^3 = 20 \implies x^3 = \frac{5}{3} \implies x = 1.1856\ldots$$

$$S = 6(1.1856)^2 + \frac{20}{1.1856} = 8.43 + 16.87 = 25.3 \text{ m}^2 \text{ (1 d.p.)}$$

(c) Justify the minimum.

$$\frac{d^2S}{dx^2} = 12 + \frac{40}{x^3} > 0 \quad \text{for all } x > 0$$

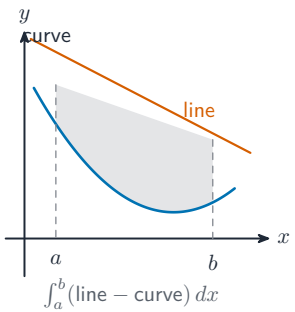
- Positive second derivative everywhere on the domain: the stationary point is a minimum ✓.
- Check: at $x = 1.1856$, $\frac{d^2S}{dx^2} = 12 + 40/(5/3) = 12 + 24 = 36 > 0$ ✓.

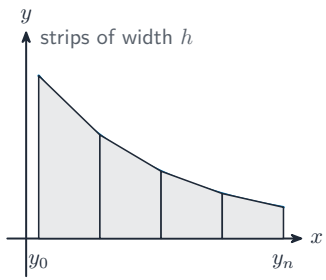
Integration: areas and the trapezium rule

Evaluate **definite integrals** with the fundamental theorem:

$$\int_a^b y\,dx = F(b) - F(a) \quad \text{where } F'(x) = y$$

The definite integral is the **area under the curve** for $y \geq 0$. For a region bounded by a curve and a line, integrate (line − curve) between their intersections, or subtract the two areas – always draw a sketch to decide which way round.





Area between = ∫(line − curve); the trapezium rule estimates ∫curve with straight strips.

Worked example – area between curve and line (adapted from Edexcel IAL WMA12 June 2025 Q3). C: $y = 3x + \dfrac{16}{x^2} - 8$ $(x > 0)$ with $P(1, 11)$ and $Q(4, 5)$ on C; find the area between C and the line segment PQ.

- Line PQ: gradient $\dfrac{5-11}{4-1} = -2$, so $y - 11 = -2(x-1) \implies y = -2x + 13$.
- Check Q: $-8 + 13 = 5$ ✓.
- Area under the line: $\displaystyle\int_1^4 (-2x + 13)\,dx = \left[-x^2 + 13x\right]_1^4 = 36 - 12 = 24$.
- Area under the curve: $\displaystyle\int_1^4 \left(3x + 16x^{-2} - 8\right) dx = \left[\tfrac{3}{2}x^2 - \tfrac{16}{x} - 8x\right]_1^4 = (24 - 4 - 32) - (\tfrac{3}{2} - 16 - 8) = 10.5$.
- Region area: $24 - 10.5 = \mathbf{13.5}$ square units.
- Check: the line is above the curve throughout (at $x = 2$: line 9, curve $3(2) + 4 - 8 = 2$), so subtracting curve from line is the right way round ✓.

The trapezium rule approximates an area with n strips of width h:

$$\int_a^b y\,dx \approx \frac{h}{2}\left[\tfrac{1}{2}(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1}\right] \quad \text{or} \quad \frac{h}{2}\left[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})\right]$$

More strips (or a straighter curve) improve the estimate; a curve that bends upwards (**convex**, U-shaped like x^2) leaves every chord **above** it, so the rule **overestimates**, while a curve that bends downwards (**concave**, like $\sqrt{x}$ or a circle arc) leaves chords **below** it, so the rule **underestimates**.

Worked example – trapezium rule with a table. Estimate $\displaystyle\int_1^3 \frac{1}{x}\,dx$ with 4 strips.

- Four strips over $[1, 3]$ give $h = 0.5$ and five values of $y = \dfrac{1}{x}$: at $x = 1,\ 1.5,\ 2,\ 2.5,\ 3$ the values are $1,\ 0.6667,\ 0.5,\ 0.4,\ 0.3333$.

$$\int_1^3 \frac{1}{x}\,dx \approx \frac{0.5}{2}\left[1 + 0.3333 + 2(0.6667 + 0.5 + 0.4)\right] = 0.25 \times 4.4667 = 1.12$$

- The true value is $\ln 3 = 1.099$ (3 s.f.). The estimate is an **overestimate**: $y = 1/x$ is convex (it bends upwards), so every straight strip sits above the curve ✓.

Checklist before you call this unit done

- Structure a proof; use exhaustion over complete cases; disprove with one counterexample.
- Apply the factor and remainder theorems, including $(ax - b)$ divisors, and carry out algebraic division with quotient and remainder.
- Find a circle's centre and radius from either form; use the three circle properties.
- Use a_n, S_n and S_∞ for arithmetic and geometric series (both sum-formula proofs), Σ notation, and logs to find n.
- Expand $(a + bx)^n$ with binomial coefficients, and match given coefficients.
- Solve log equations with the three laws, checking domains; give intersections exactly as $(\log_a b, c)$ when asked.
- Reduce trigonometric equations with $\tan = \sin / \cos$ and $\sin^2 + \cos^2 = 1$; solve in degrees or radians as given, listing all interval solutions.
- Find and classify stationary points; justify with a sign change or second derivative; set up modelling minima in one variable.
- Evaluate definite integrals; find areas under curves and between curve and line; apply the trapezium rule from a table and state the direction of its error.