7.4 Integrating hyperbolic and inverse functions

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS integration by parts • completing the square

Integrating hyperbolic and inverse functions

Objective

Integrate hyperbolic expressions, use the four standard surd/rational forms, complete the square under surds, and integrate inverse functions by parts (FP3 4.1–4.4).

1 Key knowledge and worked examples

Standard forms. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2+a^2} = \tfrac{1}{a}\arctan\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2-a^2} = \tfrac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$. Complete the square first when the quadratic has a linear term. For $\int(\text{inverse function})\,dx$, integrate by parts with the inverse function as u and $1\cdot dx$ as dv.

Worked example – both techniques (WFM03 June 2025 Q2). (i) $4x^2 + 8x + 9 = 4\left((x+1)^2 + \frac{5}{4}\right)$, so $\displaystyle\int \frac{dx}{\sqrt{4x^2+8x+9}} = \tfrac{1}{2}\operatorname{arsinh}\frac{2x+2}{\sqrt{5}} + c$. (ii) $\displaystyle\int \operatorname{arcosh} 3x\,dx = x\operatorname{arcosh} 3x - \int \frac{3x\,dx}{\sqrt{9x^2-1}} = x\operatorname{arcosh} 3x - \tfrac{1}{3}\sqrt{9x^2-1} + c$.

2 Practice

2.1 Find $\displaystyle\int \sinh^3 x \cosh x\,dx$. [3]

2.2 Evaluate $\displaystyle\int_0^1 \frac{dx}{x^2+4}$. [4]

2.3 Find $\displaystyle\int \frac{dx}{x^2-9}$. [4]

2.4 Find $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WFM03 June 2025 Q2.)* Determine

(i) $\displaystyle\int \frac{dx}{\sqrt{4x^2+8x+9}}$

(ii) $\displaystyle\int \operatorname{arcosh} 3x \, dx$. [7]

3.2 *(Original.)* Evaluate $\int_0^1 \operatorname{arsinh} x \, dx$, giving your answer in exact form. [4]