

6.3 Euler’s relation, de Moivre’s theorem and complex roots

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS Euler’s relation • de Moivre’s theorem • modulus • argument

Euler’s relation, de Moivre’s theorem and complex roots

Objective

Use Euler’s relation and de Moivre’s theorem to expand multiple-angle identities and powers of trigonometric functions, prove the theorem for integer n , and find all n th roots of a complex number in exact polar form (FP2 3.1–3.2).

1 Key knowledge and worked examples

Core relations. $e^{i\theta} = \cos \theta + i \sin \theta$, so $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. De Moivre: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$, provable for positive integers n by induction (the inductive step multiplies by $\cos \theta + i \sin \theta$ and expands with the addition formulae); negative integers follow by taking reciprocals.

Worked example – powers to multiple angles (WFM02 June 2025 Q9). The imaginary part of $(\cos \theta + i \sin \theta)^5$ is $5c^4s - 10c^2s^3 + s^5$; substituting $c^2 = 1 - s^2$ gives $\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$. Setting $5\theta = \pi$ yields $16s^4 - 20s^2 + 5 = 0$, so $s^2 = \frac{5 \pm \sqrt{5}}{8}$: $\sin \frac{\pi}{5} = \sqrt{\frac{5 - \sqrt{5}}{8}}$ (the smaller root, since $\sin \frac{\pi}{5} < \sin \frac{3\pi}{5}$).

Worked example – fifth roots (WFM02 June 2025 Q1). For $z^5 = -2\sqrt{2} - 2\sqrt{6}i$: modulus $\sqrt{8 + 24} = 4\sqrt{2}$, argument $-\frac{2\pi}{3}$ (third quadrant, tangent $\sqrt{3}$). So $z = \sqrt{2}e^{i(-\frac{2\pi}{15} + \frac{2\pi k}{5})}$, $k = 0, 1, 2, 3, 4$ – five roots spaced $\frac{2\pi}{5}$ apart on $|z| = \sqrt{2}$. Exact polar form only.

2 Practice

2.1 Use de Moivre’s theorem to express $\cos 3\theta$ as a polynomial in $\cos \theta$. [3]

2.2 Express $\sin^3 \theta$ in the form $a \sin \theta + b \sin 3\theta$. [3]

2.3 Solve $z^3 = 8i$, giving the roots in the form $re^{i\theta}$ with exact θ . [4]

3 Exam-style questions

3.1 (*Original.*) (a) Show that $\cos^4 \theta = \frac{1}{8}(3 + 4 \cos 2\theta + \cos 4\theta)$. (b) Hence evaluate $\int_0^{\pi/2} \cos^4 \theta \, d\theta$. [7]

3.2 (*Edexcel IAL WFM02 June 2025 Q1.*) (a) Express $-2\sqrt{2}-2\sqrt{6}i$ in the form $r(\cos \theta + i \sin \theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. (b) Hence solve $z^5 = -2\sqrt{2} - 2\sqrt{6}i$, giving your answers in the form $re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$. [8]
