7.6 Vectors and further matrix algebra

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS vector product • scalar triple product • eigenvalue • eigenvector

Vectors and further matrix algebra

Objective

Compute vector and scalar triple products with their geometric meanings, convert between line and plane forms, find point-to-plane and skew-line distances, and handle 3×3 determinants, inverses and eigenvalues (FP3 5.1–5.3, 6.4–6.8).

1 Key knowledge and worked examples

Vectors. $\mathbf{a} \times \mathbf{b}$ via the determinant with $\mathbf{i}, \mathbf{j}, \mathbf{k}$; its modulus is the parallelogram area. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is the parallelepiped volume (zero $\iff$ coplanar). Planes: $\mathbf{r} \cdot \mathbf{n} = p$ (normal from $\mathbf{b} \times \mathbf{c}$ in the parametric form). Lines: $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. Shortest distance between skew lines with directions $\mathbf{b}_1, \mathbf{b}_2$ through $\mathbf{a}_1, \mathbf{a}_2$: $\dfrac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$.

Matrices. det by row/column expansion; $\mathbf{A}^{-1} = \frac{1}{\det}(\text{adjugate})$. Eigenvector check row by row: each component of $\mathbf{Mv}$ equals λ times the matching component of $\mathbf{v}$ – not λ itself. For symmetric matrices, normalised eigenvectors form an orthogonal $\mathbf{P}$ with $\mathbf{P}^T\mathbf{AP}$ diagonal.

Worked example – eigenvalues from a given eigenvector (WFM03 June 2025 Q3). $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$, eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Row 1: $1 - 2 + 4 = 3 = 3(1)$, so $\lambda = 3$. Row 2: $3 + 2a + b = 3(2) = 6$; row 3: $a + 2 + b = 3(1) = 3$. Solving: $a = 2$, $b = -1$. The trace (2) and determinant (-6) then force the other eigenvalues 1 and -2, with eigenvectors $(-1, 4, 1)$ and $(-1, 1, 1)$.

2 Practice

2.1 Find $(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$. [2]

2.2 Find the volume of the parallelepiped spanned by $\mathbf{a} = (1, 0, 2)$, $\mathbf{b} = (3, 1, 0)$ and $\mathbf{c} = (0, 2, 1)$. [3]

2.3 Evaluate the determinant of $\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 5 & 2 & -2 \end{pmatrix}$. [3]

2.4 Find the eigenvalues and eigenvectors of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and write down an orthogonal matrix $\mathbf{P}$ that diagonalises it. [4]

3 Exam-style questions

3.1 *(Edexcel IAL WFM03 June 2025 Q8, condensed.)* The plane Π_1 has equation $\mathbf{r} = (4, 2, -1) + \lambda(7, 5, -6) + \mu(-3, 1, 2)$. (a) Determine $(7\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}) \times (-3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. (b) Hence show that the equation of Π_1 can be written as $\mathbf{r} \cdot (8\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}) = p$, stating the value of p. Given that the line of intersection of Π_1 with the plane $\Pi_2 : x - y + z = 7$ has direction $(13, 3, -10)$ and passes through $(\frac{13}{5}, -\frac{17}{5}, 1)$, and that the line l_2 passes through $A(2, 1, 3)$ and $B(3, 0, 2)$, (c) determine the shortest distance between the two lines. [7]

3.2 *(Edexcel IAL WFM03 June 2025 Q3, condensed.)* $\mathbf{M} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix}$, where a and b are constants, has eigenvector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. (a) Determine the corresponding eigenvalue. (b) Hence determine a and b. (c) Given that the other eigenvalues of $\mathbf{M}$ are 1 and -2, determine an eigenvector corresponding to $\lambda = 1$. [6]

Solutions

2.1 $\mathbf{i}\big((-3)(-2)-(1)(4)\big)-\mathbf{j}\big((2)(-2)-(1)(1)\big)+\mathbf{k}\big((2)(4)-(-3)(1)\big)=2\mathbf{i}+5\mathbf{j}+11\mathbf{k}$.

2.2 $\mathbf{b}\times\mathbf{c}=(1\cdot1-0\cdot2,\ 0\cdot0-3\cdot1,\ 3\cdot2-1\cdot0)=(1,-3,6)$; $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=1+0+12=13$. Volume 13 cubic units.

2.3 Expanding along the first row: $2\big((-1)(-2)-4\cdot2\big)-1\big(0\cdot(-2)-4\cdot5\big)+3\big(0\cdot2-(-1)\cdot5\big)=2(-6)+20+15=23$.

2.4 $\det\begin{pmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{pmatrix}=(2-\lambda)^2-1=0$ gives $\lambda=3$ or 1. $\lambda=3$: $(\mathbf{M}-3\mathbf{I})\mathbf{v}=\mathbf{0}$ gives $v_1=v_2$, so $\begin{pmatrix}1\\1\end{pmatrix}$; $\lambda=1$: $v_1=-v_2$, so $\begin{pmatrix}1\\-1\end{pmatrix}$. Normalising, $\mathbf{P}=\dfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ (or with columns swapped/negated), and $\mathbf{P}^T\mathbf{A}\mathbf{P}=\operatorname{diag}(3,1)$.

3.1 (a) $(7,5,-6)\times(-3,1,2)=\big(5\cdot2-(-6)(1),\ (-6)(-3)-7\cdot2,\ 7\cdot1-5\cdot(-3)\big)=(16,4,22)=2(8,2,11)$. (b) The normal is $(8,2,11)$; substituting the anchor point: $p=4\cdot8+2\cdot2+(-1)\cdot11=32+4-11=25$. (c) l_2 has direction $\mathbf{b}_2=(3,0,2)-(2,1,3)=(1,-1,-1)$. Common normal: $(13,3,-10)\times(1,-1,-1)=\big(3\cdot(-1)-(-10)(-1),\ (-10)(1)-13\cdot(-1),\ 13\cdot(-1)-3\cdot1\big)=(-13,3,-16)$, with $|(-13,3,-16)|=\sqrt{434}$. Offset between anchors: $(2,1,3)-(\frac{13}{5},-\frac{17}{5},1)=(-\frac{3}{5},\frac{22}{5},2)$. Distance $=\dfrac{|\frac{39}{5}+\frac{66}{5}-32|}{\sqrt{434}}=\dfrac{11}{\sqrt{434}}$.

3.2 (a) Row 1: $\mathbf{Mv}$'s first component is $1-2+4=3$, and $3=\lambda\cdot1$, so $\lambda=3$. (b) Row 2: $3+2a+b=3\times2=6$; row 3: $a+2+b=3\times1=3$. Subtracting: $a=2$; then $b=-1$. (c) With $a=2$, $b=-1$: $(\mathbf{M}-\mathbf{I})\mathbf{v}=\mathbf{0}$ has rows $(0,-1,4)$ and $(3,1,-1)$ (and $(2,1,-2)$). From row 1: $v_2=4v_3$; row 2: $3v_1+v_2-v_3=0$, so with $v_2=4v_3$, $3v_1=-3v_3$ and $v_1=-v_3$. Taking $v_3=1$: eigenvector $(-1,4,1)$.

Local review guidance: cross products with the correct cyclic signs (middle term negated); plane constant from an anchor point; eigen checks compare component-wise with λv_j; distances projected onto the common normal with exact surds. This is authored classroom guidance.