7.5 Reduction formulae, arc length and surfaces of revolution

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS reduction formula • arc length • surface of revolution

Reduction formulae, arc length and surfaces of revolution

Objective

Derive and iterate reduction formulae by parts, and compute arc lengths and areas of surfaces of revolution exactly in cartesian form (FP3 4.5–4.6).

1 Key knowledge and worked examples

Formulas. Arc length $s = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$; surface area $S = 2\pi \int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ (curve rotated 2π about the x-axis). Reduction formulae: integrate by parts so that the leftover integral can be rewritten in terms of I_n and I_{n-1} (or I_{n-2}) – expand surds like $\sqrt{f(x)} = \frac{f(x)}{\sqrt{f(x)}}$ to expose them.

Worked example – building one (WFM03 June 2025 Q5). With $I_n = \int_1^6 \frac{x^n}{\sqrt{3x-2}}dx$: parts with $u = x^n$, $v = \frac{2}{3}\sqrt{3x-2}$ gives $I_n = \frac{8\cdot 6^n - 2}{3} - \frac{2n}{3}(3I_n - 2I_{n-1})$, so $(3+6n)I_n = 4nI_{n-1} + 8\cdot 6^n - 2$. Seeding $I_0 = 2$, $I_1 = 6$: $I_2 = \frac{334}{15}$ and $I_3 = \frac{3322}{35}$.

2 Practice

2.1 Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$, giving an exact answer. [4]

2.2 The curve $y = \cosh x$, $0 \le x \le \ln 2$, is rotated through 2π radians about the x-axis. Find, exactly, the area of the curved surface generated. [4]

2.3 Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$. (a) Show that, for $n \ge 2$, $nI_n = (n-1)I_{n-2}$. (b) Hence find I_4. [5]

3 Exam-style questions

3.1 *(Edexcel IAL WFM03 June 2025 Q5.)* $I_n = \int_1^6 x^n(3x-2)^{-1/2}\,dx$, $n \ge 0$. (a) Show that, for $n \ge 1$, $(3+6n)I_n = 4nI_{n-1} + 8\times 6^n - 2$. (b) Use the reduction formula in part (a) to determine the exact value of $\int_1^6 x^3(3x-2)^{-1/2}\,dx$. [9]

3.2 *(Edexcel IAL WFM03 June 2025 Q7a.)* The curve with equation $y = \cos 2x$, $0 \leq x \leq \frac{\pi}{4}$, is rotated through 2π radians about the x-axis. Show that the area of the curved surface generated is $S = 2\pi \int_0^{\pi/4} \cos 2x \sqrt{1 + 4\sin^2 2x}\, dx$. [3]

Solutions

2.1 $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$, so with $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}dx$: $s = \int_0^4 \sqrt{1 + \frac{9}{4}x}\,dx = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{8}{27}\left(10^{3/2} - 1\right)$.

2.2 $\frac{dy}{dx} = \sinh x$ and $1 + \sinh^2 x = \cosh^2 x$, so

$$S = 2\pi \int_0^{\ln 2} \cosh^2 x\,dx = 2\pi\left[\tfrac{1}{2}\sinh x \cosh x + \tfrac{x}{2}\right]_0^{\ln 2}$$

(from $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$). With $\sinh(\ln 2) = \frac{3}{4}$, $\cosh(\ln 2) = \frac{5}{4}$: $S = 2\pi\left(\frac{15}{32} + \frac{1}{2}\ln 2\right)$, i.e. $\frac{15\pi}{16} + \pi \ln 2$.

2.3 (a) Parts with $u = \sin^{n-1} x$, $dv = \sin x\,dx$: $I_n = (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x\,dx = (n-1)(I_{n-2} - I_n)$ (the boundary term vanishes at both limits), so $nI_n = (n-1)I_{n-2}$. (b) $I_0 = \frac{\pi}{2}$, $I_2 = \frac{1}{2}I_0 = \frac{\pi}{4}$, $I_4 = \frac{3}{4}I_2 = \frac{3\pi}{16}$.

3.1 (a) $v = \frac{2}{3}\sqrt{3x-2}$ (check by differentiating). Parts:

$$I_n = \left[\tfrac{2}{3}x^n\sqrt{3x-2}\right]_1^6 - \tfrac{2n}{3}\int_1^6 x^{n-1}\sqrt{3x-2}\,dx$$

The boundary term: $\frac{2}{3}\left(6^n \cdot 4 - 1\right) = \frac{8 \cdot 6^n - 2}{3}$. The integral: $\int_1^6 \frac{x^{n-1}(3x-2)}{\sqrt{3x-2}}dx = 3I_n - 2I_{n-1}$. So $I_n = \frac{8 \cdot 6^n - 2}{3} - \frac{2n}{3}(3I_n - 2I_{n-1})$, which rearranges to $(3+6n)I_n = 4nI_{n-1} + 8 \cdot 6^n - 2$. (b) $I_0 = \frac{2}{3}[\sqrt{3x-2}]_1^6 = \frac{2}{3}(4-1) = 2$; $n = 1$: $9I_1 = 4(2) + 48 - 2 = 54$, so $I_1 = 6$; $n = 2$: $15I_2 = 8(6) + 8(36) - 2 = 334$, so $I_2 = \frac{334}{15}$; $n = 3$: $21I_3 = 12 \cdot \frac{334}{15} + 8(216) - 2 = \frac{1336}{5} + 1726 = \frac{9966}{5}$, so $I_3 = \frac{9966}{105} = \frac{3322}{35}$.

3.2 $\frac{dy}{dx} = -2\sin 2x$, so $\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + 4\sin^2 2x}$ and $S = 2\pi \int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = 2\pi \int_0^{\pi/4} \cos 2x\sqrt{1 + 4\sin^2 2x}\,dx$ (no modulus needed: $\cos 2x \geq 0$ on $[0, \frac{\pi}{4}]$).

Local review guidance: parts orientation chosen so the leftover integral re-expresses in I_n, I_{n-1}; boundary values at both limits; surface integrals carry the y factor and the surd from $\frac{dy}{dx}$, with exact evaluations. This is authored classroom guidance.