7.4 Integrating hyperbolic and inverse functions

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS integration by parts • completing the square

Integrating hyperbolic and inverse functions

Objective

Integrate hyperbolic expressions, use the four standard surd/rational forms, complete the square under surds, and integrate inverse functions by parts (FP3 4.1–4.4).

1 Key knowledge and worked examples

Standard forms. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arcosh}\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2+a^2} = \tfrac{1}{a}\arctan\frac{x}{a} + c$, $\displaystyle\int \frac{dx}{x^2-a^2} = \tfrac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$. Complete the square first when the quadratic has a linear term. For $\int(\text{inverse function})\,dx$, integrate by parts with the inverse function as u and $1\cdot dx$ as dv.

Worked example – both techniques (WFM03 June 2025 Q2). (i) $4x^2 + 8x + 9 = 4\left((x+1)^2 + \frac{5}{4}\right)$, so $\displaystyle\int \frac{dx}{\sqrt{4x^2+8x+9}} = \tfrac{1}{2}\operatorname{arsinh}\frac{2x+2}{\sqrt{5}} + c$. (ii) $\displaystyle\int \operatorname{arcosh} 3x\,dx = x\operatorname{arcosh} 3x - \int \frac{3x\,dx}{\sqrt{9x^2-1}} = x\operatorname{arcosh} 3x - \tfrac{1}{3}\sqrt{9x^2-1} + c$.

2 Practice

2.1 Find $\displaystyle\int \sinh^3 x \cosh x\,dx$. [3]

2.2 Evaluate $\displaystyle\int_0^1 \frac{dx}{x^2+4}$. [4]

2.3 Find $\displaystyle\int \frac{dx}{x^2-9}$. [4]

2.4 Find $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WFM03 June 2025 Q2.)* Determine

(i) $\displaystyle\int \frac{dx}{\sqrt{4x^2+8x+9}}$

(ii) $\displaystyle\int \operatorname{arcosh} 3x \, dx$. [7]

3.2 *(Original.)* Evaluate $\int_0^1 \operatorname{arsinh} x \, dx$, giving your answer in exact form. [4]

Solutions

2.1 With $u = \sinh x$: $\frac{1}{4}\sinh^4 x + c$.

2.2 $\left[\frac{1}{2}\arctan\frac{x}{2}\right]_0^1 = \frac{1}{2}\arctan\frac{1}{2}$.

2.3 $\frac{1}{2\cdot 3}\ln\left|\dfrac{x-3}{x+3}\right| + c = \frac{1}{6}\ln\left|\dfrac{x-3}{x+3}\right| + c$.

2.4 $\operatorname{arsinh}\dfrac{x}{3} + c \ \left(= \ln\left(x + \sqrt{x^2+9}\right) + c'\right)$.

3.1 (i) $4x^2 + 8x + 9 = (2x+2)^2 + 5$, so with $u = 2x+2$, $dx = \frac{du}{2}$: $\frac{1}{2}\displaystyle\int \frac{du}{\sqrt{u^2+5}} = \frac{1}{2}\operatorname{arsinh}\frac{2x+2}{\sqrt{5}} + c$. (ii) By parts, $u = \operatorname{arcosh} 3x$, $dv = dx$: $x\operatorname{arcosh} 3x - \displaystyle\int x\cdot\frac{3}{\sqrt{9x^2-1}}dx = x\operatorname{arcosh} 3x - \frac{1}{3}\sqrt{9x^2-1} + c$ (because $\dfrac{d}{dx}\sqrt{9x^2-1} = \dfrac{9x}{\sqrt{9x^2-1}}$).

3.2 By parts: $[x\operatorname{arsinh} x]_0^1 - \displaystyle\int_0^1 \frac{x\,dx}{\sqrt{1+x^2}} = \ln(1+\sqrt{2}) - \left[\sqrt{1+x^2}\right]_0^1 = \ln(1+\sqrt{2}) - \sqrt{2} + 1$.

Local review guidance: square completed correctly under the surd (or linear substitution identified); by parts with the inverse function chosen as u; exact forms with the constant of integration on indefinite integrals. This is authored classroom guidance.