7.3 Differentiating hyperbolic and inverse functions

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS inverse function • chain rule

Differentiating hyperbolic and inverse functions

Objective

Differentiate hyperbolic expressions by the chain, product and quotient rules, and inverse trigonometric/hyperbolic functions by implicit inversion, respecting domains (FP3 3.1–3.2).

1 Key knowledge and worked examples

Derivatives. $\frac{d}{dx}\sinh x = \cosh x$, $\frac{d}{dx}\cosh x = \sinh x$, $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$; $\frac{d}{dx}\operatorname{arsinh} x = \frac{1}{\sqrt{1+x^2}}$, $\frac{d}{dx}\operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$, $\frac{d}{dx}\operatorname{artanh} x = \frac{1}{1-x^2}$, $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$. Where a domain like $x > 0$ is given, use it to resolve $\sqrt{x^2} = x$.

Worked example – inverse derivatives on a restricted domain (WFM03 June 2025 Q4). For $y = \operatorname{arsinh} x + \operatorname{arsinh}\frac{1}{x}$, $x > 0$: $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + \frac{-1/x^2}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+x^2}} - \frac{1}{x\sqrt{1+x^2}} = \frac{x-1}{x\sqrt{1+x^2}}$. The stationary point is $x = 1$, where $y = 2\operatorname{arsinh} 1 = 2\ln(1+\sqrt{2})$.

2 Practice

2.1 Differentiate $x\sinh 2x$. [4]

2.2 Differentiate $\cosh(3x^2 + 1)$. [3]

2.3 Differentiate $x^2 \operatorname{artanh} x$. [3]

2.4 Show that $\dfrac{d}{dx}(e^x \cosh x) = e^{2x}$. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WFM03 June 2025 Q4.)* $y = \operatorname{arsinh} x + \operatorname{arsinh} \dfrac{1}{x}$, $x > 0$. (a) Show that $\dfrac{dy}{dx} = \dfrac{x-1}{x\sqrt{1+x^2}}$. (b) Hence determine the exact value of y for which $\dfrac{dy}{dx} = 0$, giving your answer as a simplified natural logarithm. [6]

3.2 *(Original.)* (a) Differentiate $\operatorname{arcosh} 2x$. (b) Find the equation of the tangent to $y = \operatorname{arcosh} 2x$ at the point where $x = 1$, giving exact values. [6]

Solutions

2.1 Product rule: $\sinh 2x + 2x\cosh 2x$.

2.2 Chain rule: $6x\sinh(3x^2+1)$.

2.3 $2x\,\mathrm{artanh}\,x + \dfrac{x^2}{1-x^2}$.

2.4 $\dfrac{d}{dx}(e^x\cosh x) = e^x\cosh x + e^x\sinh x = e^x(\cosh x + \sinh x) = e^x\cdot e^x = e^{2x}$, since $\cosh x + \sinh x \equiv e^x$.

3.1 (a) $\dfrac{d}{dx}\,\mathrm{arsinh}\,x = \dfrac{1}{\sqrt{1+x^2}}$ and, by the chain rule, $\dfrac{d}{dx}\,\mathrm{arsinh}\,\dfrac{1}{x} = \dfrac{-1/x^2}{\sqrt{1+1/x^2}}$. For $x > 0$, $\sqrt{1+1/x^2} = \dfrac{\sqrt{1+x^2}}{x}$, so the second term is $-\dfrac{1}{x\sqrt{1+x^2}}$ and $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}\left(1-\frac{1}{x}\right) = \dfrac{x-1}{x\sqrt{1+x^2}}$. (b) $\dfrac{dy}{dx} = 0$ exactly at $x=1$: $y = \mathrm{arsinh}\,1 + \mathrm{arsinh}\,1 = 2\ln(1+\sqrt{2})$.

3.2 (a) $\dfrac{d}{dx}\,\mathrm{arcosh}\,2x = \dfrac{2}{\sqrt{4x^2-1}}$ (chain rule on $\sqrt{(2x)^2-1}$). (b) At $x=1$: $y = \mathrm{arcosh}\,2 = \ln(2+\sqrt{3})$ and the gradient is $\dfrac{2}{\sqrt{3}}$. Tangent: $y - \ln(2+\sqrt{3}) = \dfrac{2}{\sqrt{3}}(x-1)$.

Local review guidance: correct inverse derivative including the inner-function factor; the $x > 0$ simplification of $\sqrt{x^2}$ credited only when justified; exact values as logarithms or surds, never decimals. This is authored classroom guidance.