7.2 The ellipse and the hyperbola

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS ellipse • hyperbola • eccentricity • directrix

The ellipse and the hyperbola

Objective

Use the standard equations and parametric forms of the ellipse and hyperbola, their eccentricity/focus/directrix relations, and tangent conditions (FP3 2.1–2.4).

1 Key knowledge and worked examples

Standard facts. Ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ($x=a\cos t$, $y=b\sin t$): $b^2=a^2(1-e^2)$, foci $(\pm ae,0)$, directrices $x=\pm\dfrac{a}{e}$. Hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ ($x=a\sec t$, $y=b\tan t$; right branch $x=a\cosh t$, $y=b\sinh t$): $b^2=a^2(e^2-1)$. Tangent condition for $y=mx+c$: $c^2=a^2m^2+b^2$ (ellipse) or $c^2=a^2m^2-b^2$ (hyperbola). Focus property: $PS=e\cdot PN$ at every point.

Worked example – eccentricity, tangent, intercepts (WFM03 June 2025 Q9, condensed). For $\dfrac{x^2}{64}-\dfrac{y^2}{49}=1$: $49=64(e^2-1)$ gives $e=\frac{\sqrt{113}}{8}$, foci $(\pm\sqrt{113},0)$. At $P(8\sec\theta,7\tan\theta)$, $\dfrac{dy}{dx}=\dfrac{7\sec\theta}{8\tan\theta}$ and the tangent is $\frac{x}{8}\sec\theta-\frac{y}{7}\tan\theta=\sec^2\theta-\tan^2\theta=1$, meeting the y-axis at $Q=(0,-7\cot\theta)$.

2 Practice

2.1 For the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$, find the eccentricity, the coordinates of the foci, and the equations of the directrices. [3]

2.2 For the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$, find the eccentricity, the foci and the directrices. [3]

2.3 Find the equations of the tangents to $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ with gradient $\frac{1}{2}$. [4]

2.4 Show that the point $(4\sec\theta, 3\tan\theta)$ lies on $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ for all values of θ. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WFM03 June 2025 Q9a–c, condensed.)* The hyperbola H has equation $\dfrac{x^2}{64} - \dfrac{y^2}{49} = 1$ and eccentricity e. (a) Show that $e = \dfrac{\sqrt{113}}{8}$. (b) Use calculus to show that the tangent to H at $P(8\sec\theta, 7\tan\theta)$ has equation $\frac{x}{8}\sec\theta - \frac{y}{7}\tan\theta = 1$. (c) Write down the coordinates of the point Q where this tangent meets the y-axis. [7]

3.2 *(Original.)* The hyperbola K has equation $\dfrac{x^2}{49} - \dfrac{y^2}{32} = 1$. (a) Find its eccentricity and the coordinates of its foci. (b) Verify that the point $(7\sec\theta, 4\sqrt{2}\tan\theta)$ lies on K for all θ. [5]

Solutions

2.1 $9 = 25(1-e^2)$ gives $e = \frac{4}{5}$. Foci $(\pm ae, 0) = (\pm 4, 0)$; directrices $x = \pm\frac{a}{e} = \pm\frac{25}{4}$.

2.2 $16 = 9(e^2-1)$ gives $e^2 = \frac{25}{9}$, $e = \frac{5}{3}$. Foci $(\pm 5, 0)$; directrices $x = \pm\frac{9}{5}$.

2.3 With $m = \frac{1}{2}$: $c^2 = a^2m^2 + b^2 = 4\cdot\frac{1}{4} + 3 = 4$, so $c = \pm 2$ and the tangents are $y = \frac{1}{2}x \pm 2$. (Substituting either into the ellipse gives a quadratic with zero discriminant.)

2.4 $\dfrac{(4\sec\theta)^2}{16} - \dfrac{(3\tan\theta)^2}{9} = \sec^2\theta - \tan^2\theta = 1$ for all θ ✓.

3.1 (a) $a^2 = 64$, $b^2 = 49$: $49 = 64(e^2-1)$, so $e^2 = \frac{113}{64}$ and $e = \frac{\sqrt{113}}{8}$. (b) $x = 8\sec\theta$, $y = 7\tan\theta$: $\dfrac{dy}{dx} = \dfrac{7\sec^2\theta}{8\sec\theta\tan\theta}$. The tangent: $y - 7\tan\theta = \dfrac{7\sec\theta}{8\tan\theta}(x - 8\sec\theta)$; multiplying through by $\frac{8\tan\theta}{7}$ and collecting, $\frac{x}{8}\sec\theta - \frac{y}{7}\tan\theta = \sec^2\theta - \tan^2\theta = 1$. (c) At $x = 0$: $-\frac{y}{7}\tan\theta = 1$, so $Q = (0, -7\cot\theta)$.

3.2 (a) $a^2 = 49$, $b^2 = 32$: $32 = 49(e^2-1)$, so $e^2 = \frac{81}{49}$, $e = \frac{9}{7}$; foci $(\pm ae, 0) = (\pm 9, 0)$. (b) $\dfrac{(7\sec\theta)^2}{49} = \sec^2\theta$ and $\dfrac{(4\sqrt{2}\tan\theta)^2}{32} = \frac{32\tan^2\theta}{32} = \tan^2\theta$, so the left side is $\sec^2\theta - \tan^2\theta = 1$ ✓.

Local review guidance: correct b^2–e relation for the curve type (sign!); foci and directrices placed with ae and a/e; tangents from $\frac{dy}{dx}$ at the parametric point, or the $c^2 = a^2m^2 \pm b^2$ condition for given gradients. This is authored classroom guidance.