7.1 Hyperbolic functions and equations

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS hyperbolic functions • exponential form

Hyperbolic functions and equations

Objective

Use the exponential definitions of cosh, sinh and tanh to derive identities, convert mixed equations into quadratics in e^x, and give exact logarithmic solutions (FP3 1.1).

1 Key knowledge and worked examples

Definitions. $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\tanh x = \dfrac{\sinh x}{\cosh x}$. Core identities: $\cosh^2 x - \sinh^2 x \equiv 1$, $\cosh 2x \equiv \cosh^2 x + \sinh^2 x \equiv 2\cosh^2 x - 1$, $\sinh 2x \equiv 2\sinh x \cosh x$. Every mixed equation $a\cosh x + b\sinh x = c$ becomes a quadratic in e^x; reject roots that make $e^x \leq 0$ and check survivors.

Worked example – identity then equation (WFM03 June 2025 Q1). From the definitions, $\cosh 6x + \cosh 4x = \frac{1}{2}(e^{6x} + e^{-6x}) + \frac{1}{2}(e^{4x} + e^{-4x}) = \frac{1}{2}(e^{5x} + e^{-5x})(e^x + e^{-x}) = 2\cosh 5x \cosh x$. So $\cosh 6x + \cosh 4x = 8\cosh x$ collapses to $\cosh x(2\cosh 5x - 8) = 0$, giving $\cosh 5x = 4$ and $x = \pm\frac{1}{5}\ln(4 + \sqrt{15})$.

2 Practice

2.1 Express $6\cosh x - 2\sinh x$ in the form $Ae^x + Be^{-x}$, and hence find its minimum value. [4]

2.2 Solve $\cosh 2x = 2$, giving your answers as natural logarithms in simplest form. [3]

2.3 Solve $5\cosh x + 3\sinh x = 4$. [4]

2.4 Solve $\tanh x = \frac{3}{5}$, giving your answer as a natural logarithm. [3]

3 Exam-style questions

3.1 *(Edexcel IAL WFM03 June 2025 Q1.)* (a) Use the definition of $\cosh x$ in terms of exponentials to show that $\cosh 6x + \cosh 4x = 2\cosh 5x \cosh x$. (b) Hence determine the exact values of x for which $\cosh 6x + \cosh 4x = 8\cosh x$, giving your answers in terms of natural logarithms in simplest form. [6]

3.2 *(Original.)* (a) Use the definitions of sinh and cosh to show that $\sinh 2x = 2\sinh x \cosh x$.
(b) Hence solve $2\sinh x \cosh x = 3\sinh x$, giving exact answers in terms of natural logarithms. [5]

Solutions

2.1 $6\cosh x - 2\sinh x = 3(e^x + e^{-x}) - (e^x - e^{-x}) = 2e^x + 4e^{-x}$. By AM–GM, $2e^x + 4e^{-x} \geq 2\sqrt{2e^x \cdot 4e^{-x}} = 4\sqrt{2}$, attained when $2e^x = 4e^{-x}$, i.e. $e^{2x} = 2$, $x = \frac{1}{2}\ln 2$. Minimum value $4\sqrt{2}$.

2.2 $2x = \pm \operatorname{arcosh} 2 = \pm \ln(2 + \sqrt{3})$, so $x = \pm\frac{1}{2}\ln(2 + \sqrt{3})$.

2.3 $5\cosh x + 3\sinh x = \frac{5}{2}(e^x + e^{-x}) + \frac{3}{2}(e^x - e^{-x}) = 4e^x + e^{-x}$. So $4e^x + e^{-x} = 4$ gives $4e^{2x} - 4e^x + 1 = 0$, i.e. $(2e^x - 1)^2 = 0$, hence $e^x = \frac{1}{2}$ and $x = -\ln 2$. (Check: $5\cosh(-\ln 2) + 3\sinh(-\ln 2) = 5 \cdot \frac{5}{4} - 3 \cdot \frac{3}{4} = \frac{25-9}{4} = 4$ ✓.)

2.4 $x = \operatorname{artanh}\frac{3}{5} = \frac{1}{2}\ln\dfrac{1 + \frac{3}{5}}{1 - \frac{3}{5}} = \frac{1}{2}\ln 4 = \ln 2$.

3.1 (a) LHS $= \frac{1}{2}(e^{6x} + e^{-6x} + e^{4x} + e^{-4x}) = \frac{1}{2}\left(e^{5x} + e^{-5x}\right)\left(e^x + e^{-x}\right) = 2\cosh 5x \cosh x$, pairing $e^{6x}, e^{-6x}, e^{4x}, e^{-4x}$ as products of $e^{\pm 5x}$ with $e^{\pm x}$. (b) $2\cosh 5x \cosh x = 8\cosh x$ gives $\cosh x(2\cosh 5x - 8) = 0$; since $\cosh x \geq 1$, $\cosh 5x = 4$, so $5x = \pm\ln(4 + \sqrt{15})$ and $x = \pm\frac{1}{5}\ln(4 + \sqrt{15})$.

3.2 (a) $2\sinh x \cosh x = 2 \cdot \frac{1}{2}(e^x - e^{-x}) \cdot \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}(e^{2x} - e^{-2x}) = \sinh 2x$. (b) $\sinh 2x = 3\sinh x$ factors as $\sinh x(2\cosh x - 3) = 0$: either $\sinh x = 0$ ($x = 0$), or $\cosh x = \frac{3}{2}$, giving $x = \pm\ln\left(\frac{3}{2} + \sqrt{\frac{9}{4} - 1}\right) = \pm\ln\frac{3+\sqrt{5}}{2}$. So $x = 0$ or $x = \pm\ln\frac{3+\sqrt{5}}{2}$.

Local review guidance: exponential forms written before any cancellation; quadratic in e^x solved with invalid roots rejected; final answers as simplified logarithms with both signs where the equation is even. This is authored classroom guidance.