6.6 Maclaurin–Taylor series and polar coordinates

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS Maclaurin series • Taylor series • polar coordinates • cardioid

Maclaurin–Taylor series and polar coordinates

Objective

Derive and use Maclaurin and Taylor series (including series solutions of differential equations by differentiating the equation), convert between polar and Cartesian forms, sketch the standard polar curves, and evaluate exact polar areas with $\frac{1}{2}\int r^2\,d\theta$ (FP2 6.1–6.4, 7.1).

1 Key knowledge and worked examples

Series machinery. Maclaurin: $f(x) = \sum \dfrac{f^{(n)}(0)}{n!}x^n$; Taylor about a: $f(x) = \sum \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. For a differential equation, climb the ladder by differentiating the equation itself: $y(a)$, $y'(a)$ give $y''(a)$; differentiating gives $y'''(a)$; and so on. For polar curves, $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, and the area swept from α to β is $\frac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$.

Worked example – a series solution about a non-zero centre (WFM02 June 2025 Q2). For $y'' - 4y' - 5y = e^{2x/9}$ with $y = 2$, $y' = -1$ at $x = 3$: at $x = 3$, $y'' = 4y' + 5y + e^{2/3} = 6 + e^{2/3}$, and differentiating, $y''' = 4y'' + 5y' + \frac{2}{9}e^{2x/9}$, so $y'''(3) = 19 + \frac{38}{9}e^{2/3}$. Hence $y = 2 - (x-3) + \left(3 + \frac{1}{2}e^{2/3}\right)(x-3)^2 + \left(\frac{19}{6} + \frac{19}{27}e^{2/3}\right)(x-3)^3$.

Worked example – exact polar areas. Lemniscate $r^2 = 8\cos 2\theta$: the right loop is $\frac{1}{2}\int_{-\pi/4}^{\pi/4} 8\cos 2\theta\,d\theta = 4$, so total area 8. Cardioid $r = 2(1+\cos\theta)$: $\frac{1}{2}\int_{-\pi}^{\pi} 4(1+\cos\theta)^2 d\theta = 2(2\pi + 0 + \pi) = 6\pi$.

2 Practice

2.1 Find the Maclaurin series of $\ln(1+2x)$ up to and including the term in x^3, stating the range of validity. [3]

2.2 Find the first three non-zero terms of the Maclaurin series of $\sin x \cos x$. [2]

2.3 Convert $r = 4\cos\theta$ to a Cartesian equation and describe the curve. [3]

2.4 Find the exact area enclosed by the cardioid $r = 2(1+\cos\theta)$. [3]

3 Exam-style questions

3.1 *(Original.)* Given that $\dfrac{dy}{dx} = x^2 + y$ and $y = 1$ at $x = 0$, find the Taylor series for y about $x = 0$ up to and including the term in x^3. [6]

3.2 *(Edexcel IAL WFM02 June 2025 Q2.)* Given that y satisfies $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = e^{2x/9}$ and that $y = 2$ and $\dfrac{dy}{dx} = -1$ at $x = 3$, find the Taylor series for y about $x = 3$, up to and including the term in $(x-3)^3$. [8]

Solutions

2.1 $f' = \frac{2}{1+2x}$, $f'' = \frac{-4}{(1+2x)^2}$, $f''' = \frac{16}{(1+2x)^3}$; at 0: $0, 2, -4, 16$. So $\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 - \cdots$, valid for $|2x| < 1$, i.e. $-\frac{1}{2} < x < \frac{1}{2}$.

2.2 $\sin x \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\left[2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{120} - \cdots\right] = x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \cdots$

2.3 Multiply by r: $r^2 = 4r\cos\theta$, so $x^2 + y^2 = 4x$, i.e. $(x-2)^2 + y^2 = 4$: a circle, centre $(2, 0)$, radius 2, through the pole.

2.4 $\frac{1}{2}\int_{-\pi}^{\pi} 4(1+\cos\theta)^2\,d\theta = 2\int_{-\pi}^{\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta = 2(2\pi + 0 + \pi) = 6\pi$ (since $\int_{-\pi}^{\pi}\cos^2\theta\,d\theta = \pi$).

3.1 $y'(0) = 0 + 1 = 1$. Differentiating the equation: $y'' = 2x + y'$, so $y''(0) = 0 + 1 = 1$. Again: $y''' = 2 + y''$, so $y'''(0) = 3$. The series is $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2}$. (Check at the centre: the equation gives $y''(0) = 2(0) + y'(0) = 1$, matching the series' quadratic term $\frac{1}{2}y''(0) = \frac{1}{2}$ ✓.)

3.2 At $x = 3$: $y'' = 4y' + 5y + e^{2/3} = -4 + 10 + e^{2/3} = 6 + e^{2/3}$. Differentiating the equation: $y''' = 4y'' + 5y' + \frac{2}{9}e^{2x/9}$, so $y'''(3) = 4(6 + e^{2/3}) - 5 + \frac{2}{9}e^{2/3} = 19 + \frac{38}{9}e^{2/3}$. Therefore $y = 2 - (x-3) + \frac{6+e^{2/3}}{2}(x-3)^2 + \frac{19+\frac{38}{9}e^{2/3}}{6}(x-3)^3$, i.e. $y = 2 - (x-3) + \left(3 + \frac{1}{2}e^{2/3}\right)(x-3)^2 + \left(\frac{19}{6} + \frac{19}{27}e^{2/3}\right)(x-3)^3$.

Local review guidance: derivatives obtained by differentiating the equation, not by guessing the function; factorial divisions correct term by term; polar areas use $\frac{1}{2}r^2$ inside the integral with exact trigonometric integrals. This is authored classroom guidance.