6.4 Loci, regions and z→w transformations

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS locus • perpendicular bisector • Möbius transformation

Loci, regions and z→w transformations

Objective

Describe and sketch standard loci in the Argand diagram (circles, bisectors, Apollonius circles, argument half-lines) with their interior regions, and find the image of a locus under a Möbius transformation $w = \dfrac{az+b}{cz+d}$ by making z the subject and equating moduli or arguments (FP2 3.3–3.4).

1 Key knowledge and worked examples

Locus toolkit. $|z-a| = b$: circle, centre a, radius b. $|z-a| = |z-b|$: perpendicular bisector of a and b. $|z-a| = k|z-b|$, $k \neq 1$: a circle (square and collect). $\arg(z-a) = \beta$: half-line from a at angle β. For regions, replace $=$ by $<$ or $>$ and test one convenient point. Under $w = \dfrac{az+b}{cz+d}$: make z the subject, then take moduli (or arguments) of both sides. A circle through the pole $z = -\frac{d}{c}$ maps to a line; otherwise circle to circle.

Worked example – image of a circle (WFM02 June 2025 Q3). $w = \dfrac{9i - iz}{1+z}$ ($z \neq -1$), P on $|z| = 1$. Rearranging: $z(w+i) = 9i - w$, so $z = \dfrac{9i-w}{w+i}$ and $|z| = 1$ forces $|9i - w| = |w+i|$, i.e. $u^2 + (v-9)^2 = u^2 + (v+1)^2$, so $20v = 80$ and the image is the line $v = 4$ – the perpendicular bisector of $9i$ and $-i$. Spot-check: $z = i \mapsto 5 + 4i$, $z = -i \mapsto -5 + 4i$.

2 Practice

2.1 Describe fully the locus given by $|z - 2i| = 3$. [2]

2.2 Describe fully the locus given by $|z-3| = |z+1|$. [2]

2.3 Show that the locus $|z-2| = 2|z+1|$ is a circle, and state its centre and radius. [4]

2.4 Describe the locus given by $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$. [3]

3 Exam-style questions

3.1 *(Original.)* The transformation T from the z-plane to the w-plane is given by $w = \dfrac{1}{z}$, $z \neq 0$. Show that T maps the circle $|z-2| = 2$ onto the straight line $\operatorname{Re} w = \frac{1}{4}$, and sketch this line. [5]

3.2 *(Edexcel IAL WFM02 June 2025 Q3.)* The transformation T from the z-plane to the w-plane is given by $w = \dfrac{9i - iz}{1 + z}$, $z \neq -1$. The point P represents the complex number z, where $|z| = 1$. (a) Sketch the locus of P as z varies. (b) Given that the image under T of the locus of P, where $z \neq -1$, is the line l, determine, in simplest form, a Cartesian equation for l. (c) Write down the image under T of the point $z = i$ and verify that it lies on l. [9]

Solutions

2.1 Circle, centre $2i$ (the point $(0, 2)$), radius 3.

2.2 Perpendicular bisector of the points 3 and -1: the vertical line $\operatorname{Re} z = 1$.

2.3 With $z = x+iy$: $(x-2)^2+y^2 = 4\left[(x+1)^2+y^2\right]$, so $x^2-4x+4+y^2 = 4x^2+8x+4+4y^2$, i.e. $3x^2+12x+3y^2 = 0$, so $x^2+4x+y^2 = 0$, i.e. $(x+2)^2+y^2 = 4$: circle, centre $(-2, 0)$, radius 2. (Check: $z = 0$ gives $|-2| = 2|1|$ ✓ and 0 lies on it.)

2.4 $\arg\left(\frac{z-1}{z+1}\right) = \arg(z-1) - \arg(z+1)$ is the angle at z subtended by the segment from -1 to 1. Equal to $\frac{\pi}{2}$: the semicircle $x^2+y^2 = 1$, $y > 0$, excluding the endpoints ± 1.

3.1 $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$. Then $|\frac{1}{w} - 2| = 2$ gives $|1-2w| = 2|w|$. Squaring: $1-2w-2\bar{w}+4|w|^2 = 4|w|^2$, so $w+\bar{w} = \frac{1}{2}$, i.e. $\operatorname{Re} w = \frac{1}{4}$ – a vertical line in the w-plane, $\frac{1}{4}$ unit to the right of the imaginary axis. (Check: $z = 4$ is on the circle, and $z = 4 \mapsto w = \frac{1}{4}$ ✓.)

3.2 (a) Unit circle, centre the origin. (b) $z = \dfrac{9i-w}{w+i}$, so $|9i-w| = |w+i| \cdot |z| = |w+i|$. With $w = u+iv$: $9i-w = -u+i(9-v)$, so $u^2+(9-v)^2 = u^2+(v+1)^2$, giving $81-18v = 2v+1$, i.e. $20v = 80$: l is $v = 4$. (c) $z = i$: $w = \dfrac{9i-i^2}{1+i} = \dfrac{1+9i}{1+i} = \dfrac{(1+9i)(1-i)}{2} = \dfrac{10+8i}{2} = 5+4i$, whose imaginary part is 4, so it lies on l ✓.

Local review guidance: z made the subject before any modulus work; modulus equation expanded in u, v; line/circle description with geometry named (bisector, centre, radius). This is authored classroom guidance.