6.3 Euler's relation, de Moivre's theorem and complex roots

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS Euler's relation • de Moivre's theorem • modulus • argument

Euler's relation, de Moivre's theorem and complex roots

Objective

Use Euler's relation and de Moivre's theorem to expand multiple-angle identities and powers of trigonometric functions, prove the theorem for integer n, and find all nth roots of a complex number in exact polar form (FP2 3.1–3.2).

1 Key knowledge and worked examples

Core relations. $e^{i\theta} = \cos\theta + i\sin\theta$, so $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$. De Moivre: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, provable for positive integers n by induction (the inductive step multiplies by $\cos\theta + i\sin\theta$ and expands with the addition formulae); negative integers follow by taking reciprocals.

Worked example – powers to multiple angles (WFM02 June 2025 Q9). The imaginary part of $(\cos\theta + i\sin\theta)^5$ is $5c^4s - 10c^2s^3 + s^5$; substituting $c^2 = 1 - s^2$ gives $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. Setting $5\theta = \pi$ yields $16s^4 - 20s^2 + 5 = 0$, so $s^2 = \frac{5\pm\sqrt{5}}{8}$: $\sin\frac{\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}$ (the smaller root, since $\sin\frac{\pi}{5} < \sin\frac{3\pi}{5}$).

Worked example – fifth roots (WFM02 June 2025 Q1). For $z^5 = -2\sqrt{2} - 2\sqrt{6}i$: modulus $\sqrt{8+24} = 4\sqrt{2}$, argument $-\frac{2\pi}{3}$ (third quadrant, tangent $\sqrt{3}$). So $z = \sqrt{2}\,e^{i\left(-\frac{2\pi}{15} + \frac{2\pi k}{5}\right)}$, $k = 0, 1, 2, 3, 4$ – five roots spaced $\frac{2\pi}{5}$ apart on $|z| = \sqrt{2}$. Exact polar form only.

2 Practice

2.1 Use de Moivre's theorem to express $\cos 3\theta$ as a polynomial in $\cos\theta$. [3]

2.2 Express $\sin^3\theta$ in the form $a\sin\theta + b\sin 3\theta$. [3]

2.3 Solve $z^3 = 8i$, giving the roots in the form $re^{i\theta}$ with exact θ. [4]

3 Exam-style questions

3.1 *(Original.)* (a) Show that $\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$. (b) Hence evaluate $\displaystyle\int_0^{\pi/2} \cos^4\theta \, d\theta$. [7]

3.2 *(Edexcel IAL WFM02 June 2025 Q1.)* (a) Express $-2\sqrt{2}-2\sqrt{6}i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. (b) Hence solve $z^5 = -2\sqrt{2} - 2\sqrt{6}i$, giving your answers in the form $re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$. [8]

Solutions

2.1 Real part of $(c+is)^3$: $c^3 - 3cs^2 = c^3 - 3c(1-c^2) = 4\cos^3\theta - 3\cos\theta$.

2.2 From $\sin 3\theta = 3s - 4s^3$ (imaginary part of $(c+is)^3$): $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$.

2.3 $8i = 8e^{i\pi/2}$, so $z = 2e^{i(\pi/6+2\pi k/3)}$, $k = 0,1,2$: $2e^{i\pi/6}$, $2e^{i5\pi/6}$, $2e^{i3\pi/2}$. (Check: the product of the roots is $8e^{i(\pi/6+5\pi/6+3\pi/2)} = 8e^{i5\pi/2} = 8i$ ✓, matching $z_1z_2z_3 = -(-8i)$.)

3.1 (a) $\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$. (Or: real part of $(\cos\theta + i\sin\theta)^4$ is $c^4 - 6c^2s^2 + s^4$; substitute $c^2 = \frac{1+\cos 2\theta}{2}$, $s^2 = \frac{1-\cos 2\theta}{2}$.) (b) $\displaystyle\int_0^{\pi/2} \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)\,d\theta = \frac{1}{8}\left[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{8}\cdot\frac{3\pi}{2} = \frac{3\pi}{16}$.

3.2 (a) Modulus $\sqrt{(-2\sqrt{2})^2 + (-2\sqrt{6})^2} = \sqrt{32} = 4\sqrt{2}$. Both components negative, and $\tan\theta = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$ gives a reference angle $\frac{\pi}{3}$, so $\theta = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$: $4\sqrt{2}\left(\cos\frac{2\pi}{3}\cdot(-1)\ldots\right)$, i.e. $4\sqrt{2}(\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3}))$. (b) $z = (4\sqrt{2})^{1/5}e^{i\left(\frac{-2\pi/3+2\pi k}{5}\right)} = \sqrt{2}\,e^{i\left(-\frac{2\pi}{15}+\frac{2\pi k}{5}\right)}$, $k = 0,1,2,3,4$, i.e. arguments $-\frac{2\pi}{15}$, $\frac{4\pi}{15}$, $\frac{10\pi}{15} = \frac{2\pi}{3}$, $\frac{16\pi}{15} - 2\pi = -\frac{14\pi}{15}$, $\frac{22\pi}{15} - 2\pi = -\frac{8\pi}{15}$ (all in $-\pi < \theta \le \pi$): $z \in \left\{\sqrt{2}e^{-2\pi i/15}, \sqrt{2}e^{4\pi i/15}, \sqrt{2}e^{2\pi i/3}, \sqrt{2}e^{-14\pi i/15}, \sqrt{2}e^{-8\pi i/15}\right\}$.

Local review guidance: modulus and argument found exactly with the correct quadrant; roots carry the $\frac{2\pi k}{n}$ spread and land inside the stated argument range; power identities simplified with $c^2 + s^2 = 1$ substitutions. This is authored classroom guidance.