6.1 Inequalities with the modulus sign

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS modulus • case split • strict inequality

Inequalities with the modulus sign

Objective

Solve inequalities involving moduli by case splitting from the definition, by squaring both sides, and under a modulus-valued denominator, reporting dead branches and reusing an earlier result when the question says "hence" (FP2 1.1).

1 Key knowledge and worked examples

Two safe moves. Case split from $|f| = f$ where $f \geq 0$ and $-f$ where $f < 0$, intersecting each branch with its own condition; and square $|A| > |B|$ to $A^2 > B^2$ (valid for strict and weak inequalities because both sides are non-negative). Never multiply or divide by an expression of unfixed sign – case-split on its sign instead.

Worked example – one branch dies. Solve $|x^2 - 1| > 2(x + 1)$. Where $x^2 \geq 1$: $x^2 - 2x - 3 > 0$, so $x < -1$ or $x > 3$ (keep only $x \leq -1$ or $x \geq 1$). Where $x^2 < 1$: $1 - x^2 > 2x + 2$ becomes $(x + 1)^2 < 0$ – impossible. Solution: $x < -1$ or $x > 3$. Report the dead branch: it is part of the argument.

Worked example – modulus inside a denominator (WFM02 June 2025 Q8). With $y = \dfrac{15x}{|x| + 4}$ and line $y = x - 2$: solve $x - 2 > \dfrac{15x}{|x| + 4}$. The denominator is always positive, so multiply through. For $x \geq 0$: $x^2 - 13x - 8 > 0$ gives $x > \frac{13+\sqrt{201}}{2}$. For $x < 0$: $(x+1)(x+8) < 0$ gives $-8 < x < -1$. Then "hence" solve $|x-2| > \left|\frac{15x}{|x|+4}\right|$: squaring gives the union of part (a) with $(x-2)(|x|+4) < -15x$, whose branches are $0 \leq x < \frac{\sqrt{321}-17}{2}$ and all $x < 0$.

2 Practice

2.1 Solve $|x - 3| < 5$. [2]

2.2 Solve $|2x + 1| \geq 7$. [2]

2.3 Solve $|x - 1| > 2x$. [3]

2.4 Solve $|x^2 - 4| \leq 3x$. [4]

3 Exam-style questions

3.1 *(Original.)* Use algebra to solve $|x + 2| > |2x - 1|$. [4]

3.2 *(Edexcel IAL WFM02 June 2025 Q8.)* The curve C has equation $y = \dfrac{15x}{|x|+4}$ and the line l has equation $y = x - 2$. (a) Use algebra to determine the exact values of x for which $x - 2 > \dfrac{15x}{|x|+4}$. (b) Hence use algebra to determine the exact values of x for which $|x-2| > \left|\dfrac{15x}{|x|+4}\right|$. [10]

Solutions

2.1 $-5 < x - 3 < 5$, so $-2 < x < 8$.

2.2 $2x + 1 \geq 7$ or $2x + 1 \leq -7$: $x \geq 3$ or $x \leq -4$.

2.3 Where $x \geq 1$: $x - 1 > 2x$ gives $x < -1$, which contradicts $x \geq 1$ – no solutions. Where $x < 1$: $1 - x > 2x$ gives $x < \frac{1}{3}$. Solution: $x < \frac{1}{3}$.

2.4 Where $x^2 \geq 4$ ($x \leq -2$ or $x \geq 2$): $x^2 - 3x - 4 \leq 0$, so $-1 \leq x \leq 4$; keeping the case gives $2 \leq x \leq 4$. Where $x^2 < 4$ ($-2 < x < 2$): $x^2 + 3x - 4 \geq 0$, so $x \leq -4$ or $x \geq 1$; keeping the case gives $1 \leq x < 2$. Union: $1 \leq x \leq 4$. (Check $x = 4$: $|12| = 12 = 3(4)$ ✓; $x = 0.5$: $|-3.75| > 1.5$ ×, outside.)

3.1 Squaring: $(x + 2)^2 > (2x - 1)^2$, so $x^2 + 4x + 4 > 4x^2 - 4x + 1$, i.e. $3x^2 - 8x - 3 < 0$. Factorising $(3x + 1)(x - 3) < 0$ gives $-\frac{1}{3} < x < 3$.

3.2 (a) The denominator $|x| + 4 > 0$ always. For $x \geq 0$: $(x - 2)(x + 4) > 15x \Rightarrow x^2 - 13x - 8 > 0$, roots $\frac{13 \pm \sqrt{201}}{2}$, so $x > \frac{13 + \sqrt{201}}{2}$ (the other root is negative). For $x < 0$: $(x - 2)(4 - x) > 15x \Rightarrow -x^2 - 9x - 8 > 0 \Rightarrow (x + 1)(x + 8) < 0$, so $-8 < x < -1$. Solution: $-8 < x < -1$ or $x > \frac{13 + \sqrt{201}}{2}$. (b) Both sides non-negative, so squaring preserves the inequality: $(x - 2)^2 (|x| + 4)^2 > 225x^2$ factorises as $\big[(x - 2)(|x| + 4) - 15x\big]\big[(x - 2)(|x| + 4) + 15x\big] > 0$, i.e. the union of part (a) with $(x - 2)(|x| + 4) < -15x$. New branch, $x \geq 0$: $x^2 + 17x - 8 < 0$ gives $0 \leq x < \frac{\sqrt{321} - 17}{2}$. New branch, $x < 0$: $-x^2 + 21x - 8 < 0 \Rightarrow x^2 - 21x + 8 > 0$, whose roots $\frac{21 \pm \sqrt{409}}{2}$ are both positive, so every $x < 0$ qualifies. Solution: $x < \frac{\sqrt{321} - 17}{2}$ or $x > \frac{13 + \sqrt{201}}{2}$.

Local review guidance: cases named with their defining condition; valid multiplication/squaring step with sign justified; exact surd roots carried through; "hence" reuses part (a) rather than restarting. This is authored classroom guidance.