

## 5.6 Matrices, transformations, series and induction

---

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

**Total: 36 marks**

**KEY WORDS** determinant

- area scale factor
- inverse matrix
- mathematical induction
- enlargement

### Matrices, transformations, series and induction

---

#### Objective

---

Carry out  $2 \times 2$  matrix algebra (determinants, inverses, matrix equations); identify and combine geometrical transformations; use the determinant as an area scale factor; sum finite series with the standard formulae; and complete induction proofs (FP1 5, 6, 7, 8).

## 1 Key knowledge and worked examples

**Inverse and equations.**  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ ;  $\mathbf{M}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ ;  
 $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ . A matrix equation  $\mathbf{M}^{-1} = \lambda\mathbf{M} + \mu\mathbf{I}$  is solved by comparing all four entries.

**Worked example – entry comparison (WFM01 June 2025 Q1).**  $\mathbf{M} = \begin{pmatrix} 1 & a \\ 3 & -5 \end{pmatrix}$ ,  $a \neq -\frac{5}{3}$ :  $\mathbf{M}^{-1} = \frac{1}{-(5+3a)} \begin{pmatrix} -5 & -a \\ -3 & 1 \end{pmatrix}$ . Given  $\mathbf{M}^{-1} = 2\mathbf{M} + 8\mathbf{I} = \begin{pmatrix} 10 & 2a \\ 6 & -2 \end{pmatrix}$ , the  $(2, 1)$  entries give  $\frac{-3}{-(5+3a)} = 6 \Rightarrow 5 + 3a = \frac{1}{2}$ , so  $a = -\frac{3}{2}$ ; the remaining entries agree.

**Worked example – reading transformations off matrices (WFM01 June 2025 Q2).**  $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  sends  $\mathbf{i} \rightarrow (0, -1)^T$ ,  $\mathbf{j} \rightarrow (1, 0)^T$ : rotation  $90^\circ$  clockwise about  $O$ .  $\mathbf{Q} = 5\mathbf{I}$ : enlargement, factor 5, centre  $O$ .  $\mathbf{R}$  (reflection in  $y = -x$ )  $= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$  maps  $(4, 3)^T \rightarrow (-3, -4)^T = A$ . If  $\mathbf{Q}$  after  $\mathbf{P}$  sends  $B$  to  $A$ , then  $B = \mathbf{P}^{-1}\mathbf{Q}^{-1}A = \frac{1}{5}\mathbf{P}^{-1}(-3, -4)^T = \frac{1}{5}(4, -3)^T = (0.8, -0.6)$ .

**Worked example – a sum to  $2n$  (WFM01 June 2025 Q8a) and an induction (Q8b).**  $\sum_{r=1}^{2n} (2r^2 - 1) = 2 \cdot \frac{(2n)(2n+1)(4n+1)}{6} - 2n = \frac{4n(4n^2 + 3n - 1)}{3} = \frac{4}{3}n(n+1)(4n-1)$  (using  $4n^2 + 3n - 1 = (n+1)(4n-1)$ ). Induction for  $\sum_{r=1}^n r(3r-2)^2 = \frac{1}{4}n^2(n+1)(9n-7)$ : true at  $n = 1$ ; adding the  $(k+1)$ th term  $(k+1)(3k+1)^2$  to  $\frac{1}{4}k^2(k+1)(9k-7)$  gives  $\frac{1}{4}(k+1)(9k^3 + 29k^2 + 24k + 4)$ , and  $(k+1)(k+2)(9k+2)$  expands to the same cubic – the target form for  $n = k+1$ .

## 2 Practice

**2.1** For  $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , find  $\mathbf{AB}$ ,  $\det \mathbf{A}$  and  $\mathbf{A}^{-1}$ . [4]

**2.2** Describe fully the single geometrical transformation represented by (a)  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ; (b)  $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ ; (c)  $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ ; (d)  $\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$ . [4]

---



---



---

**2.3** A triangle of area 2 is transformed by  $\mathbf{M}$  with  $\det \mathbf{M} = -7$ . Find the area of the image and state what the sign tells you. [2]

---



---

**2.4** Use standard formulae to show that  $\sum_{r=1}^n (r+1)^2 = \frac{n}{6}(2n^2 + 9n + 13)$ . [4]

---



---



---



---

**2.5** Prove by induction that  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  for all positive integers  $n$ . [3]

---



---



---

### 3 Exam-style questions

---

**3.1** (*Adapted from Edexcel IAL WFM01 June 2025 Q1.*)  $\mathbf{M} = \begin{pmatrix} 1 & a \\ 2 & -3 \end{pmatrix}$ ,  $a \neq -\frac{3}{2}$ . (a) Determine  $\mathbf{M}^{-1}$  in terms of  $a$ . (b) Given that  $\mathbf{M}^{-1} = 2\mathbf{M} + 4\mathbf{I}$ , determine the value of  $a$ . [6]

**3.2** (*Adapted from Edexcel IAL WFM01 June 2025 Q8.*) (a) Using standard summation formulae, show that  $\sum_{r=1}^{2n} (2r^2 - 1) = \frac{4}{3}n(n+1)(4n-1)$ . (b) Prove by induction that  $\sum_{r=1}^n r(3r-2)^2 = \frac{1}{4}n^2(n+1)(9n-7)$ . (c) Hence solve the equation  $8 \sum_{r=1}^n r(3r-2)^2 = 15 \sum_{r=1}^{2n} (2r^2 - 1)$ . [13]

## Solutions

**2.1**  $\mathbf{AB} = \begin{pmatrix} 1 & -3 \\ 4 & -2 \end{pmatrix}$ ;  $\det \mathbf{A} = 10$ ;  $\mathbf{A}^{-1} = \frac{1}{10} \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$ .

**2.2** (a) Reflection in  $y = x$ ; (b) enlargement, scale factor 3, centre  $O$ ; (c) stretch, factor 2, parallel to the  $x$ -axis; (d) rotation  $30^\circ$  anticlockwise about  $O$ .

**2.3** Image area  $= 2 \times 7 = 14$ ; the negative determinant means the transformation reverses orientation (includes a reflection).

**2.4**  $\sum (r^2 + 2r + 1) = \frac{n(n+1)(2n+1)}{6} + n(n+1) + n = \frac{n(n+1)(2n+1) + 6n(n+1) + 6n}{6} = \frac{n(2n^2 + 3n + 1 + 6n + 6 + 6)}{6} = \frac{n(2n^2 + 9n + 13)}{6}$  ✓ (check  $n = 1$ :  $4$  and  $\frac{24}{6} = 4$ ).

**2.5**  $n = 1$ : both sides  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  ✓. If true for  $k$ :  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$  ✓. True for  $n = 1$  and  $n = k \Rightarrow n = k + 1$ ; by induction, true for all positive integers.

**3.1** (a)  $\det \mathbf{M} = -3 - 2a$ , so  $\mathbf{M}^{-1} = \frac{1}{-(3+2a)} \begin{pmatrix} -3 & -a \\ -2 & 1 \end{pmatrix}$ . (b)  $2\mathbf{M} + 4\mathbf{I} = \begin{pmatrix} 6 & 2a \\ 4 & -2 \end{pmatrix}$ .

Entry (2, 2):  $\frac{1}{-(3+2a)} = -2 \Rightarrow 3 + 2a = \frac{1}{2} \Rightarrow a = -\frac{5}{4}$ . Every other entry confirms it:

(2, 1):  $\frac{-2}{-1/2} = 4$  ✓; (1, 1):  $\frac{-3}{-1/2} = 6$  ✓; (1, 2):  $\frac{-a}{-1/2} = 2a$  ✓ (identically true).

**3.2** (a)  $2 \sum_1^{2n} r^2 - 2n = \frac{2(2n)(2n+1)(4n+1)}{6} - 2n = \frac{2n(2n+1)(4n+1) - 6n}{3} = \frac{2n(8n^2 + 6n - 2)}{3} = \frac{4n(4n^2 + 3n - 1)}{3} = \frac{4}{3}n(n+1)(4n-1)$  (since  $4n^2 + 3n - 1 = (n+1)(4n-1)$ ). (b) Base  $n = 1$ :  $1 = \frac{1}{4} \cdot 1 \cdot 2 \cdot 2$  ✓. Step: adding  $(k+1)(3k+1)^2$  to  $\frac{1}{4}k^2(k+1)(9k-7)$  gives  $\frac{1}{4}(k+1)[9k^3 + 29k^2 + 24k + 4] = \frac{1}{4}(k+1)(k+1)(k+2)(9k+2)$ , the target for  $n = k + 1$ ; close with the induction statement. (c)  $8 \cdot \frac{1}{4}n^2(n+1)(9n-7) = 15 \cdot \frac{4}{3}n(n+1)(4n-1)$  – cancel  $n(n+1)$  (positive):  $2n(9n-7) = 20(4n-1) \Rightarrow 9n^2 - 47n + 10 = 0 \Rightarrow (9n-2)(n-5) = 0$ , so  $n = 5$  ( $n$  a positive integer).

Local review guidance: inverse formula and entry-by-entry comparison; exact values; induction with base case, inductive step and closing sentence. This is authored classroom guidance.