5.5 Parabola and rectangular hyperbola

Name: _____ Class: _____ Date: _____ **Total: 25 marks**

KEY WORDS parabola • rectangular hyperbola • focus • directrix • normal

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Objective

Use the Cartesian and parametric equations of the parabola $y^2 = 4ax$ and rectangular hyperbola $xy = c^2$, their focus–directrix property, and tangents/normals by Cartesian differentiation (FP1 4.1–4.4).

1 Key knowledge and worked examples

Standard curves. Parabola: $y^2 = 4ax$, parametric $(at^2, 2at)$; focus $(a, 0)$, directrix $x = -a$. Rectangular hyperbola: $xy = c^2$, parametric $(ct, \frac{c}{t})$. Gradients by Cartesian differentiation: $y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y} = \frac{1}{t}$ at parameter t; $xy = c^2 \Rightarrow \frac{dy}{dx} = -\frac{c^2}{x^2} = -\frac{1}{t^2}$.

Worked example – normal to a hyperbola (WFM01 June 2025 Q4). For $xy = 81$ at $P(9t, \frac{9}{t})$: $\frac{dy}{dx} = -\frac{81}{x^2} = -\frac{1}{t^2}$, so the normal has gradient t^2 and equation $y - \frac{9}{t} = t^2(x - 9t)$, i.e. $ty = t^3x + 9(1 - t^4)$. At the y-axis, $A = (0, \frac{9(1-t^4)}{t})$. For $t = \frac{1}{3}$: $P = (3, 27)$, $A = (0, \frac{80}{3})$ and triangle OPA has area $\frac{1}{2} \times 3 \times \frac{80}{3} = 40$.

Worked example – focus-directrix distances (WFM01 June 2025 Q9). P on a parabola is 5 units from the focus, hence 5 units from the directrix ($PF = PM$). Points Q, R on the directrix have $PQ = PR = 10$. Right triangle PQM: $QM = \sqrt{100 - 25} = 5\sqrt{3}$, and M lies between Q and R, so $QR = 10\sqrt{3}$.

Worked example – tangent to a parabola (original). $y^2 = 12x$ at parameter $t = 2$ gives the point $(at^2, 2at) = (12, 12)$ and gradient $\frac{1}{t} = \frac{1}{2}$: tangent $y - 12 = \frac{1}{2}(x - 12)$, i.e. $x - 2y + 12 = 0$. Check: $144 = 12 \times 12$ ✓ on the curve.

2 Practice

2.1 The parabola C has equation $y^2 = 8x$. Write down the coordinates of its focus and the equation of its directrix, and the parametric coordinates of a general point. [3]

2.2 For $y^2 = 20x$ at the point with $t = 3$, find the gradient of the tangent and its equation. [3]

2.3 The rectangular hyperbola $xy = 16$ has a point Q with $x = 8$. Find $\frac{dy}{dx}$ at Q and the equation of the normal at Q. [4]

2.4 A point P on the parabola $y^2 = 4ax$ is 7 units from its focus. Find the distance from P to the directrix and the x-coordinate of P. [3]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WFM01 June 2025 Q4.)* The rectangular hyperbola H has equation $xy = 36$; the point $P(6t, \frac{6}{t})$, $t \neq 0$, lies on H. (a) Use calculus to show that the normal to H at P has equation $ty = t^3x + 6(1 - t^4)$. (b) The normal meets the y-axis at A; find the exact coordinates of A in terms of t. (c) Given $t = \frac{1}{2}$, find the exact area of triangle OPA. [8]

3.2 *(Adapted from Edexcel IAL WFM01 June 2025 Q9.)* Figure: a parabola with focus F, directrix d, and a point P on the curve. The points Q and R lie on d with $PQ = PR = 12$, and P is 6 units from F. Determine the exact length QR. [4]

Solutions

2.1 $4a = 8 \Rightarrow a = 2$: focus $(2, 0)$; directrix $x = -2$; general point $(2t^2, 4t)$.

2.2 Point $(45, 30)$; gradient $\frac{1}{t} = \frac{1}{3}$. Tangent: $y - 30 = \frac{1}{3}(x - 45) \Rightarrow x - 3y + 45 = 0$.

2.3 $Q = (8, 2)$: $\frac{dy}{dx} = -\frac{16}{x^2} = -\frac{1}{4}$, so the normal has gradient 4: $y - 2 = 4(x - 8) \Rightarrow y = 4x - 30$.

2.4 By the locus definition the distance to the directrix is also 7. Since both distances are horizontal, $PF = PM = x_P + a = 7$, so $x_P = 7 - a$.

3.1 (a) $\frac{dy}{dx} = -\frac{36}{x^2}$; at $x = 6t$: $-\frac{36}{36t^2} = -\frac{1}{t^2}$; normal gradient t^2: $y - \frac{6}{t} = t^2(x - 6t) \Rightarrow ty = t^3x + 6(1 - t^4)$ ✓. (b) $A = (0, \frac{6(1-t^4)}{t})$. (c) $t = \frac{1}{2}$: $P = (3, 12)$ and $A = (0, \frac{6(1-1/16)}{1/2}) = (0, 6 \cdot \frac{15}{16} \cdot 2) = (0, \frac{45}{4})$. Area $= \frac{1}{2} \times 3 \times \frac{45}{4} = \frac{135}{8}$.

3.2 $PF = PM = 6$; $QM = \sqrt{144 - 36} = \sqrt{108} = 6\sqrt{3}$; $QR = 2 \times 6\sqrt{3} = 12\sqrt{3}$.

Local review guidance: gradient from Cartesian differentiation, not parametric rules; exact coordinates and areas; focus-directrix arguments phrased through the locus definition. This is authored classroom guidance.