5.4 Numerical solution of equations

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS interval bisection • linear interpolation • Newton–Raphson method - • continuity

Numerical solution of equations

Objective

Locate roots by interval bisection, linear interpolation and the Newton–Raphson process, showing all stages of working (FP1 3.1). Functions stay within P1/P2 content.

1 Key knowledge and worked examples

Three locators. A sign change of continuous f on $[a, b]$ brackets a root. **Bisection** halves the bracket each time; **linear interpolation** uses the chord: $\beta \approx a - \dfrac{(b-a)f(a)}{f(b)-f(a)}$; **Newton–Raphson** (booklet) iterates $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. Calculator-only answers score zero – every substitution is shown.

Worked example – bisection (WFM01 June 2025 Q3i). $f(x) = x^2 + 5 - 8^{5x}$ has a single root α in $[0, 1]$: $f(0) = 4 > 0$; $f(0.5) = 5.25 - 8^{2.5} = -175.77 < 0$ so $\alpha \in [0, 0.5]$; midpoint 0.25: $f(0.25) = 5.0625 - 8^{1.25} = -8.392 < 0$, so the root lies in $[0, 0.25]$ – an interval of width 0.25.

Worked example – interpolation on a transcendental (WFM01 June 2025 Q3ii). $g(x) = 3^{\sin x} - 3\cos x$: $g(4) = 2.396$, $g(5) = -0.502$. Both continuous and sign change, so a root $\beta \in [4, 5]$. One interpolation: $\beta \approx 4 + \dfrac{(5-4) \times 2.396}{2.396 - (-0.502)} = 4 + \dfrac{2.396}{2.898} = 4.827$ (4 s.f.).

Worked example – Newton–Raphson (original). $x^3 - 2x - 5 = 0$ from $x_0 = 2$: $f'(x) = 3x^2 - 2$; $x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-1}{10} = 2.1$; $x_2 = 2.1 - \frac{0.061}{11.23} = 2.0946$ (true root $2.09455\ldots$).

2 Practice

2.1 Show that $x^3 + x - 3 = 0$ has a root between 1 and 2, then perform two bisections to locate it in an interval of width 0.25. [4]

2.2 Use one linear interpolation on $[2, 3]$ for $h(x) = x + \frac{1}{x} - 3$ (take $h(2) = -0.5$, $h(3) = 0.3\dot{3}$) to estimate the root to 3 significant figures. [3]

2.3 Apply two Newton–Raphson iterations to $x^3 - 3x - 1 = 0$ starting from $x_0 = 2$, giving answers to 4 decimal places. [4]

2.4 *(Original.)* A student's bisection table shows $f(1.6) = -0.3$ and $f(1.8) = 0.2$, and concludes "root = 1.7". Give two criticisms. [2]

3 Exam-style questions

3.1 *(Original.)* $p(x) = x^4 - 3x - 1$. (a) Show $p(x) = 0$ has a root γ in $[1, 2]$. (b) Use interval bisection twice to find an interval of width 0.25 containing γ. (c) Use one linear interpolation on your interval from (b) to estimate γ to 3 significant figures. [7]

3.2 *(Adapted from Edexcel IAL WFM01 June 2025 Q3ii.)* $g(x) = x \cos \frac{x}{2} + 1$, x in radians. (a) Show that $g(x) = 0$ has a root β in the interval $[3, 4]$, making the continuity argument explicit. (b) Use linear interpolation once on $[3, 4]$ to estimate β to 4 significant figures ($g(3) = 1.2122\ldots$, $g(4) = -0.6646\ldots$; use these values). [4]

Solutions

2.1 $f(1) = -1 < 0$, $f(2) = 7 > 0$, continuous $\Rightarrow$ root in $[1, 2]$. $f(1.5) = 1.875 > 0 \Rightarrow [1, 1.5]$; $f(1.25) = 0.203125 > 0 \Rightarrow [1.25, 1.5]$ (width 0.25).

2.2 The chord from $(2, -0.5)$ to $(3, 0.3\dot{3})$ crosses zero at $x \approx 2 + \dfrac{(3-2) \times 0.5}{0.3\dot{3} + 0.5} = 2 + \dfrac{0.5}{0.8\dot{3}} = 2.60$ (3 s.f.). (Check: $h(2.6) = 2.6 + 0.385 - 3 = -0.015$; true root $\frac{3+\sqrt{5}}{2} = 2.618$.)

2.3 $f'(x) = 3x^2 - 3$. $f(2) = 1$, $f'(2) = 9$: $x_1 = 2 - \frac{1}{9} = 1.8\dot{8}$. $f(1.8\dot{8}) = 6.740 - 5.667 - 1 = 0.073$; $f'(1.8\dot{8}) = 7.70$: $x_2 = 1.8\dot{8} - \frac{0.073}{7.70} = 1.8794$ (true root $1.87939\ldots$).

2.4 A bisection locates an interval, never an exact value – the root is only known to lie in $(1.6, 1.8)$ after one halving; and the midpoint 1.7 is an estimate whose accuracy is unquantified without further iterations or a stated tolerance.

3.1 (a) $p(1) = -3 < 0$, $p(2) = 9 > 0$, p continuous $\Rightarrow$ root $\gamma \in [1, 2]$. (b) $p(1.5) = 5.0625 - 5.5 = -0.4375 < 0 \Rightarrow [1.5, 2]$; $p(1.75) = 9.3789 - 6.25 = 3.129 > 0 \Rightarrow [1.5, 1.75]$ (width 0.25). (c) $\gamma \approx 1.5 + \dfrac{0.25 \times 0.4375}{0.4375 + 3.129} = 1.5 + \dfrac{0.109}{3.567} = 1.531$ (3 s.f.; true root $1.5396\ldots$ – the interpolation sits just below it).

3.2 (a) $g(3) = 3\cos 1.5 + 1 = 1.2122 > 0$, $g(4) = 4\cos 2 + 1 = -0.6646 < 0$, and g is continuous on $[3, 4]$ (a product and sum of continuous functions), so the sign change gives a root $\beta \in [3, 4]$. (b) $\beta \approx 3 + \dfrac{(4-3) \times 1.2122}{1.2122 + 0.6646} = 3 + \frac{1.2122}{1.8768} = 3.6459 \approx 3.646$ (4 s.f.; true root $3.690\ldots$ – the curve bends away from the chord here, so one interpolation is only a first estimate).

Local review guidance: each stage of working shown with substitutions visible; conclusions stated as intervals with the continuity/sign-change justification; answers to the requested accuracy only. This is authored classroom guidance.