5.3 Roots of quadratic equations

Name: _____ Class: _____ Date: _____ **Total: 29 marks**

KEY WORDS sum of roots • product of roots • symmetric expression

Roots of quadratic equations

Objective

Use $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ to evaluate symmetric expressions and form new quadratic equations with transformed roots (FP1 2.1–2.3).

1 Key knowledge and worked examples

Symmetric toolkit. $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$; $\alpha^3 + \beta^3 \equiv (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$; $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$. Never solve for the individual roots.

Worked example – a new equation without solving (WFM01 June 2025 Q6). $f(x) = 3x^2 + kx - 5$ has roots α, β: $\alpha\beta = -\frac{5}{3}$. The condition $\alpha + \beta = 9\alpha\beta$ gives $-\frac{k}{3} = -15$, so $k = 45$ and $\alpha + \beta = -15$. For new roots $\alpha^2 + \beta$ and $\alpha + \beta^2$:

- sum $= \alpha^2 + \beta^2 + \alpha + \beta = (\alpha+\beta)^2 - 2\alpha\beta + (\alpha+\beta) = 225 + \frac{10}{3} - 15 = \frac{640}{3}$
- product $= (\alpha^3 + \beta^3) + \alpha\beta + (\alpha\beta)^2 = -3450 - \frac{5}{3} + \frac{25}{9} = -\frac{31040}{9}$

New equation: $9x^2 - 1920x - 31040 = 0$.

Worked example – reciprocals. Roots $\frac{1}{\alpha}, \frac{1}{\beta}$ have sum $\dfrac{\alpha+\beta}{\alpha\beta}$ and product $\dfrac{1}{\alpha\beta}$: for $x^2 - 7x + 10$ (sum 7, product 10) the reciprocal-roots equation is $x^2 - \frac{7}{10}x + \frac{1}{10} = 0$, i.e. $10x^2 - 7x + 1 = 0$.

2 Practice

2.1 For $2x^2 - 8x + 3 = 0$ with roots α, β, find (a) $\alpha + \beta$; (b) $\alpha\beta$; (c) $\alpha^2 + \beta^2$; (d) $\frac{1}{\alpha} + \frac{1}{\beta}$. [4]

2.2 The equation $x^2 + 6x + 4 = 0$ has roots α, β. Without solving it, form a quadratic equation with roots 3α and 3β. [3]

2.3 For $x^2 - 5x + 9 = 0$ with roots α, β, find the value of $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$. [3]

2.4 *(Original.)* $x^2 + px + q = 0$ has roots α and $\alpha + 2$. Show that $p^2 - 4q = 4$ for every such pair. [3]

3 Exam-style questions

3.1 *(Original.)* $g(x) = 2x^2 - 6x + m = 0$ has roots α, β with $\alpha^2 + \beta^2 = 11$. (a) Find m. (b) Form a quadratic equation with integer coefficients whose roots are α^3 and β^3. [6]

3.2 *(Adapted from Edexcel IAL WFM01 June 2025 Q6.)* $f(x) = 3x^2 + kx - 5$ has roots α, β with $\alpha + \beta = 9\alpha\beta$. (a) State $\alpha\beta$ and find k. (b) Prove that $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$ by expanding $(\alpha + \beta)^3$. (c) Without solving $f(x) = 0$, find a quadratic equation with integer coefficients whose roots are $\alpha^2 + \beta$ and $\alpha + \beta^2$. [10]

Solutions

2.1 (a) $\alpha + \beta = 4$; (b) $\alpha\beta = \frac{3}{2}$; (c) $16 - 3 = 13$; (d) $\dfrac{4}{3/2} = \frac{8}{3}$.

2.2 New sum $= 3(\alpha+\beta) = -18$; new product $= 9\alpha\beta = 36$. Equation: $x^2+18x+36 = 0$.

2.3 $\dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \dfrac{25-18}{9} = \dfrac{7}{9}$.

2.4 Sum $2\alpha + 2 = -p$ and product $\alpha(\alpha+2) = q$. Then $p^2 = 4(\alpha+1)^2 = 4\alpha^2 + 8\alpha + 4$ and $4q = 4\alpha^2 + 8\alpha$, so $p^2 - 4q = 4$ ✓ – independent of α.

3.1 (a) $(\alpha+\beta)^2 - 2\alpha\beta = 9 - m = 11 \Rightarrow m = -2$. (b) $\alpha\beta = -1$; $\alpha^3+\beta^3 = 27 - 3(-1)(3) = 36$; $\alpha^3\beta^3 = (-1)^3 = -1$. Equation: $x^2 - 36x - 1 = 0$.

3.2 (a) $\alpha\beta = -\frac{5}{3}$; $-\frac{k}{3} = 9(-\frac{5}{3}) = -15 \Rightarrow k = 45$, $\alpha + \beta = -15$. (b) $(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 = (\alpha^3+\beta^3) + 3\alpha\beta(\alpha+\beta)$; rearrange. (c) Sum $= \frac{640}{3}$, product $= -\frac{31040}{9}$ (see worked example): $9x^2 - 1920x - 31040 = 0$.

Local review guidance: symmetric identities applied without finding individual roots; exact fractions kept until the final integer equation; new equations presented as $x^2 - (\text{sum})x + \text{product} = 0$. This is authored classroom guidance.