5.1 Complex numbers: arithmetic, Argand diagrams and polar form

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS complex number • conjugate • modulus • argument • Argand diagram

Complex numbers: arithmetic, Argand diagrams and polar form

Objective

Work confidently with complex numbers in the forms $a + bi$ and $r(\cos\theta + i\sin\theta)$: modulus, argument, conjugate, equality, sums, products and quotients, and their geometrical representation on an Argand diagram (FP1 1.1–1.3).

1 Key knowledge and worked examples

Modulus, argument, conjugate. For $z = a + bi$: $|z| = \sqrt{a^2 + b^2}$, $\bar{z} = a - bi$, and $\tan\theta = \frac{b}{a}$ with the quadrant chosen by the signs of a and b. Products multiply moduli: $|z_1 z_2| = |z_1|\,|z_2|$.

Worked example – a quotient with no fractions left. Simplify $\dfrac{4+7i}{2-3i}$: multiply numerator and denominator by the conjugate of the denominator, $\dfrac{(4+7i)(2+3i)}{2^2+3^2} = \dfrac{8+12i+14i+21i^2}{13} = \dfrac{-13+26i}{13} = -1+2i$. Check: $(2-3i)(-1+2i) = -2+4i+3i-6i^2 = 4+7i$ ✓.

Worked example – a real sum pins the imaginary part (WFM01 June 2025 Q5). $z_1 = r(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$ and z_2 has modulus 5. $|z_1 z_2| = 5r = 15$ gives $r = 3$, so $z_1 = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$. If $z_1 + z_2$ is real then $\mathrm{Im}(z_2) = \frac{3}{2}$, and $a^2 + \frac{9}{4} = 25$ gives $z_2 = \pm\frac{\sqrt{91}}{2} + \frac{3}{2}i \approx \pm 4.77 + 1.5i$ – one in the first quadrant, one in the second, both plotted 1.5 above the real axis.

2 Practice

2.1 Write each in the form $a + bi$: (a) $(3 - 2i) + (5 + 7i)$; (b) $(2 + i)(4 - 3i)$; (c) $\dfrac{5 + i}{3 - i}$. [3]

2.2 Find the modulus and argument (exact, radians) of (a) $3 + 4i$; (b) $-2 + 2i$; (c) $-1 - \sqrt{3}\,i$. [3]

2.3 Solve $x^2 + 4x + 29 = 0$, giving roots in the form $a \pm bi$. [2]

2.4 $z_1 = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ and $z_2 = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$. Find $|z_1 z_2|$ and hence write $z_1 z_2$ in polar form. [3]

2.5 On one Argand diagram plot $z = 2 + 3i$, $\bar{z}$, iz and $-z$, describing the geometry that links each pair. [3]

3 Exam-style questions

3.1 *(Original.)* The complex number w satisfies $|w| = 4$ and w has argument $\frac{2\pi}{3}$. (a) Write w exactly in the form $a + bi$. (b) Find $|w^2|$. (c) Mark w and w^2 on an Argand diagram, justifying the position of w^2 using moduli. [4]

3.2 *(Adapted from Edexcel IAL WFM01 June 2025 Q5.)* $z_1 = 3(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$ and $z_2 = a + bi$. Given that $|z_1 z_2| = 6$ and that $z_1 + z_2$ has zero imaginary part, determine the possible values of a and b. [5]

Solutions

2.1 (a) $8+5i$; (b) $8-6i+4i-3i^2 = 11-2i$; (c) $\dfrac{(5+i)(3+i)}{10} = \dfrac{15+5i+3i+i^2}{10} = \dfrac{14+8i}{10} = \frac{7}{5}+\frac{4}{5}i$.

2.2 (a) $|z| = 5$, $\arg z = \arctan\frac{4}{3} \approx 0.927$; (b) $2\sqrt{2}$, $\frac{3\pi}{4}$; (c) 2, $\tan\theta = \sqrt{3}$ with both parts negative so $\theta = -\frac{2\pi}{3}$ (or $\frac{4\pi}{3}$).

2.3 $(x+2)^2 + 25 = 0 \Rightarrow x = -2 \pm 5i$.

2.4 $|z_1 z_2| = \sqrt{2}\cdot 2 = 2\sqrt{2}$. In Cartesian form $z_1 = 1+i$ and $z_2 = -\sqrt{3}+i$, so $z_1 z_2 = -(1+\sqrt{3}) + (1-\sqrt{3})i$ and $|z_1 z_2|^2 = (1+\sqrt{3})^2 + (1-\sqrt{3})^2 = 8$, confirming $2\sqrt{2}$.

2.5 z at $(2,3)$; $\bar{z}$ at $(2,-3)$: reflection in the real axis; $iz = -3+2i$: quarter-turn anticlockwise about O; $-z = -2-3i$: half-turn about O.

3.1 (a) $w = 4(-\frac{1}{2}+\frac{\sqrt{3}}{2}i) = -2+2\sqrt{3}\,i$. (b) $|w^2| = 16$. (c) w in the second quadrant on the circle of radius 4; w^2 on the circle of radius 16 – multiplying scales the modulus (and turns the point through $2\times\frac{2\pi}{3} = \frac{4\pi}{3}$, equivalent to $-\frac{2\pi}{3}$, landing in the third quadrant).

3.2 $z_1 = 3(-\frac{\sqrt{3}}{2}+\frac{1}{2}i) = -\frac{3\sqrt{3}}{2}+\frac{3}{2}i$. Imaginary parts cancel: $b = -\frac{3}{2}$. Then $a^2+\frac{9}{4} = |z_2|^2 = (6/3)^2 = 4$, so $a = \pm\frac{\sqrt{7}}{2}$: $z_2 = \pm\frac{\sqrt{7}}{2} - \frac{3}{2}i$.

Local review guidance: conjugate used to make a denominator real; exact modulus and argument values; Argand positions justified by geometry, not just plotted. This is authored classroom guidance.