4.1 Proof by contradiction and partial fractions

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

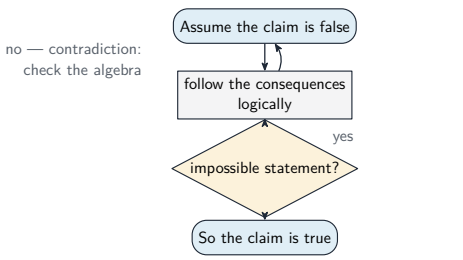
KEY WORDS proof by contradiction • partial fractions • repeated factor • improper fraction

Objective

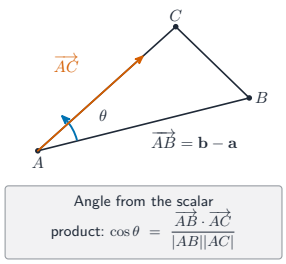
Prove statements by contradiction ($\sqrt{2}$ irrational, infinitely many primes, unfamiliar claims) and decompose rational functions into partial fractions, including repeated linear factors and improper fractions. Original teaching questions for specification items P4.1.1 and P4.2.1; the exam-style part adapts reviewed WMA14 June 2025 demand.

1 Key knowledge and worked examples

Proof by contradiction : assume the negation of the claim, argue logically until an impossibility, conclude the claim. **Partial fractions** split a rational function into sums over its denominator's factors; a repeated factor $(ax+b)^2$ contributes $\dfrac{B}{ax+b}+\dfrac{C}{(ax+b)^2}$; improper fractions divide first.



$\sqrt{2}$ irrational and the infinity of primes both run this loop



Name the assumption and the contradiction explicitly.

Worked example – contradiction. Prove there is no smallest positive rational number.

- Suppose q IS the smallest positive rational. Then $\frac{q}{2}$ is positive, rational, and $\frac{q}{2} < q$ – a smaller one exists, contradicting the assumption. ■

Worked example – repeated factor. Decompose $\dfrac{3x^2+7x+6}{x(x+1)^2}$.

$$\frac{3x^2+7x+6}{x(x+1)^2} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2} \implies 3x^2+7x+6 = A(x+1)^2+Bx(x+1)+Cx$$

- $x=0$: $A=6$. $x=-1$: $2=-C \Rightarrow C=-2$. x^2 coefficients: $3=A+B \Rightarrow B=-3$.

$$\frac{6}{x}-\frac{3}{x+1}-\frac{2}{(x+1)^2}$$

- Check at $x=1$: LHS $=16/4=4$; RHS $=6-\frac{3}{2}-\frac{1}{2}=4$ ✓.

2 Practice

2.1 Prove by contradiction that $\sqrt{3}$ is irrational. [4]

2.2 Decompose $\dfrac{7x+10}{(x+1)(x+3)}$. [3]

2.3 Decompose $\dfrac{5x+7}{(x-1)^2}$. [3]

2.4 Write $\dfrac{x^2+2x+5}{x^2+x}$ in the form $1+\dfrac{A}{x}+\dfrac{B}{x+1}$. [4]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA14 June 2025 Q10.)* Use proof by contradiction to show that $|\sin x+\cos x|<1$ for all angles x with $90^\circ<x<180^\circ$. [4]

3.2 Decompose $\dfrac{2x^2+5x+4}{(x+2)(x+1)^2}$ and verify your decomposition at $x=1$. [5]

Solutions

2.1 Suppose $\sqrt{3} = \frac{p}{q}$ in lowest terms. Then $3q^2 = p^2$, so p^2 is divisible by 3, so p is (if $p = 3m$, $p^2 = 9m^2$; conversely a prime dividing p^2 divides p). Write $p = 3m$: $3q^2 = 9m^2 \Rightarrow q^2 = 3m^2$, so q is divisible by 3 too – both divisible contradicts lowest terms. ■

2.2 $7x + 10 = A(x + 3) + B(x + 1)$: at $x = -1$, $A = \frac{3}{2}$; at $x = -3$, $B = \frac{11}{2}$. (Or compare coefficients: $A + B = 7$, $3A + B = 10$.)

2.3 With $\dfrac{5x+7}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$, multiply through: $5x + 7 = A(x - 1) + B$. Comparing x-coefficients, $A = 5$; comparing constants, $7 = -5 + B \Rightarrow B = 12$.

2.4 $\dfrac{x^2+2x+5}{x^2+x} = 1 + \dfrac{x+5}{x^2+x} = 1 + \dfrac{x+5}{x(x+1)}$; decompose $\dfrac{x+5}{x(x+1)} = \dfrac{5}{x} - \dfrac{4}{x+1}$. So $A = 5$, $B = -4$. Check at $x = 1$: $1 + 5 - 2 = 4$; original $\frac{8}{2} = 4$ ✓.

3.1 Suppose some x in $90° < x < 180°$ has $|\sin x + \cos x| \geq 1$. Squaring: $1 + \sin 2x \geq 1$, so $\sin 2x \geq 0$. But $180° < 2x < 360°$ forces $\sin 2x < 0$ – a contradiction. So $|\sin x + \cos x| < 1$ throughout. ■ Check at $x = 135°$: $|0| = 0 < 1$ ✓.

3.2 $\dfrac{2x^2+5x+4}{(x+2)(x+1)^2} = \dfrac{A}{x+2} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$: $2x^2 + 5x + 4 = A(x+1)^2 + B(x+2)(x+1) + C(x+2)$.

- $x = -2$: $8 - 10 + 4 = 2 = A$. $x = -1$: $2 - 5 + 4 = 1 = C$. x^2: $2 = A + B \Rightarrow B = 0$.

$$\frac{2}{x+2} + \frac{1}{(x+1)^2}$$

  Check at $x = 1$: LHS $= \frac{2+5+4}{(1+2)(1+1)^2} = \frac{11}{12}$; RHS $= \frac{2}{3} + \frac{1}{4} = \frac{11}{12}$ ✓.

Local review guidance: assumption of the negation stated, or correct partial-fraction identity set up; exact values; verification or the named contradiction. This is authored classroom guidance.