3.4 Exponentials, logarithms and log-graph models

Name: _____ Class: _____ Date: _____ **Total: 23 marks**

KEY WORDS model • log plot • initial value

Objective

Use $y = e^x$ and $y = \ln x$ as inverses, solve $e^{ax+b} = p$ and $\ln(ax+b) = q$, model exponential growth and decay, and recover power-law and exponential relationships from straight log plots. Original teaching questions for specification items P3.3.1 to P3.3.4; the exam-style part adapts reviewed WMA13 June 2025 demand.

1 Key knowledge and worked examples

- ln is the inverse of e^x: $\ln e^x = x$, $e^{\ln x} = x$. Solve $e^{ax+b} = p$ with $x = \frac{\ln p - b}{a}$; solve $\ln(ax+b) = q$ with $x = \frac{e^q - b}{a}$.
- **Models** : $y = Ae^{kt}$ has initial value A (at $t = 0$); $k > 0$ grows, $k < 0$ decays toward the long-run floor.
- **Log plots** : $y = ax^n$ gives $\log y = n \log x + \log a$ (plot $\log y$ vs $\log x$: gradient n, intercept $\log a$); $y = kb^x$ gives $\log y = x \log b + \log k$ (plot $\log y$ vs x).

Worked example – recover a power law (adapted from Edexcel IAL WMA13 June 2025 Q3). A $\log_{10} y$ against $\log_{10} x$ line passes through $(-3.5, 0)$ and $(0, -2)$.

- Gradient $= \dfrac{0-(-2)}{-3.5-0} = -\dfrac{4}{7}$, intercept $= -2$:
$$\log_{10} y = -\tfrac{4}{7} \log_{10} x - 2$$
- Un-log: $y = 10^{-\frac{4}{7}\log x - 2} = 10^{-2} \cdot 10^{\log x^{-4/7}} = \dfrac{1}{100} x^{-4/7}$.
- Check with $(0, -2)$: $\log x = 0 \Rightarrow x = 1$; the formula gives $y = \frac{1}{100}$ and $\log_{10} \frac{1}{100} = -2$ ✓.

Worked example – decay model (adapted from Edexcel IAL WMA13 June 2025 Q6). A bath cools as $\theta = 20 + 60e^{-kt}$ (θ in °C, t in minutes); it reaches 35 °C after 8 minutes. Find k and the time at which the temperature falls at 2 °C per minute.

- At $t = 8$: $35 = 20 + 60e^{-8k} \implies e^{-8k} = \frac{1}{4}$, so
$$k = \tfrac{1}{8} \ln 4 = 0.173 \text{ (3 s.f.)}$$
- Rate: $\dfrac{d\theta}{dt} = -60ke^{-kt}$; setting $= -2$:
$$e^{-kt} = \frac{2}{60 \times 0.173} = 0.1927 \implies t = \frac{\ln(1/0.1927)}{0.173} = 9.5 \text{ minutes (1 d.p.)}$$
- Check: $\theta(9.5) = 20 + 60e^{-0.173(9.5)} = 20 + 60(0.1924) = 31.5$ °C and $60 \times 0.173 \times 0.1924 = 2.0$ °C/min ✓. Long-run floor: 20 °C.

2 Practice

2.1 Solve $e^{2x} = 50$, giving x to 3 significant figures. [2]

2.2 Solve $\ln(3x - 2) = 4$ exactly. [2]

2.3 A population is modelled by $P = 2000e^{0.05t}$. Find P when $t = 10$ and state the model's initial value. [2]

2.4 For $y = ax^n$, a $\log y$–$\log x$ plot has gradient 3 and intercept 0.778 ($= \log 6$). Find a and n. [2]

2.5 Write $\log_{10} y = 2x + 1$ as an exponential relationship between y and x. [2]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA13 June 2025 Q3.)* A $\log_{10} y$ against $\log_{10} x$ line passes through $(-2, 1)$ and $(0, 5)$. Express y in the form px^q, giving p and q exactly. [5]

3.2 *(Adapted from Edexcel IAL WMA13 June 2025 Q6.)* The temperature of an engine is modelled by $\theta = 25 + Ae^{-kt}$ (θ in °C, t in minutes). The engine is 95 °C when switched off at $t = 0$, and 40 °C at $t = 6$.

(a) Find A and k (3 s.f.).

(b) Find the time at which the temperature is decreasing at 4 °C per minute (2 d.p.).

(c) Explain why the model cannot hold below 25 °C. [8]

Solutions

2.1 $x = \frac{1}{2}\ln 50 = 1.96$ (3 s.f.).

2.2 $3x - 2 = \mathrm{e}^4$, so $x = \dfrac{\mathrm{e}^4 + 2}{3}$.

2.3 $P = 2000\mathrm{e}^{0.5} = 3297$; initial value 2000 (at $t = 0$).

2.4 $n = 3$ and $\log a = \log 6 \Rightarrow a = 6$: $y = 6x^3$.

2.5 $y = 10^{2x+1} = 10 \times 100^x$.

3.1 Gradient $= \dfrac{5-1}{0-(-2)} = 2$; intercept 5:

$$\log_{10} y = 2\log_{10} x + 5 \implies y = 10^5\, x^2 = 100\,000\, x^2$$

$p = 100\,000$, $q = 2$. Check at $(0, 5)$: $x = 1$, $y = 10^5$, $\log y = 5$ ✓.

3.2 (a) At $t = 0$: $95 = 25 + A \Rightarrow A = 70$. At $t = 6$: $40 = 25 + 70\mathrm{e}^{-6k} \Rightarrow \mathrm{e}^{-6k} = \frac{15}{70} = \frac{3}{14}$, so

$$k = \tfrac{1}{6}\ln\tfrac{14}{3} = 0.257 \text{ (3 s.f.)}$$

(b) $\dfrac{d\theta}{dt} = -70k\mathrm{e}^{-kt} = -4$:

$$\mathrm{e}^{-kt} = \frac{4}{70 \times 0.257} = 0.2226 \implies t = \frac{\ln(1/0.2226)}{0.257} = 5.85 \text{ (2 d.p.)}$$

Check: $\theta(5.85) = 25 + 70\mathrm{e}^{-0.257(5.85)} = 25 + 70(0.2226) = 40.6$ °C, falling at $70 \times 0.257 \times 0.2226 = 4.0$ °C/min ✓. (c) As $t \to \infty$, $\theta \to 25$ °C from above: the model's floor is room temperature, so it never predicts values below 25 °C – a colder-than-room prediction needs a new model.

Local review guidance: correct log recovery or correct differentiation of the model; exact/rounded values as asked; model interpretation tied to the limit. This is authored classroom guidance.