2.1 Proof: exhaustion and counterexample

Name: _____ Class: _____ Date: _____ **Total: 20 marks**

KEY WORDS proof • proof by exhaustion • counterexample

Objective

Structure mathematical proofs, use proof by exhaustion over complete sets of cases, and disprove statements with a counterexample. Original teaching questions for specification items P2.1.1 to P2.1.3; the exam-style part adapts reviewed WMA12 June 2025 demand.

1 Key knowledge and worked examples

A **proof** starts from given assumptions, moves by logical steps, and ends with the target conclusion. Quote every assumption you use; do not assume what you are proving.

- **Proof by exhaustion** lists every possible case and checks each one. The case list must be complete – "every integer is of the form $3q$, $3q+1$ or $3q+2$" is complete; "the first few integers" is not.
- **Disproof by counterexample** needs exactly one example that breaks the claim.

Worked example – exhaustion. Prove that for every integer n, the number n^2+n is even.

- The cases split on parity, and every integer is in one case:
- n even, $n=2k$: $n^2+n=4k^2+2k=2(2k^2+k)$ – even.
- n odd, $n=2k+1$: $n^2+n=(2k+1)(2k+2)$, and $2k+2$ is even, so the product is even.
- Both cases give an even number, so n^2+n is always even. ■
- Check: $n=4$: 20 ✓; $n=5$: 30 ✓; $n=-3$: 6 ✓.

Worked example – counterexample. Disprove: "2^n+1 is prime for every positive integer n."

- Try small values: $3, 5, 9, \ldots$ – and $2^3+1=9=3\times 3$ is not prime.
- One counterexample ($n=3$) disproves the claim for all n. ■

2 Practice

2.1 Disprove by counterexample: "$n^2 + 1$ is prime for all positive integers n." [2]

2.2 Prove that the sum of three consecutive integers is divisible by 3. [3]

2.3 Prove by exhaustion that $n^2 - 1$ is divisible by 3 for every positive integer n that is not a multiple of 3. [4]

2.4 Prove that the product of two odd integers is odd. [3]

3 Exam-style questions

3.1 *(Adapted from Edexcel IAL WMA12 June 2025 Q10.)* Use algebra to prove by exhaustion that for all positive integers m that are not multiples of 3, the value of $m^2 + 3m + 2$ is always a multiple of 3. [4]

3.2 Prove by exhaustion that no perfect square ends in the digit 8. [4]

Solutions

2.1 $n = 3$: $9 + 1 = 10 = 2 \times 5$, which is not prime. One counterexample disproves the claim.

2.2 Let the integers be $(n-1)+n+(n+1) = 3n$, which is divisible by 3 for every integer n. ■

2.3 Non-multiples of 3 are $n = 3q + 1$ or $n = 3q + 2$ (complete case list).

- $n = 3q + 1$: $n^2 - 1 = 9q^2 + 6q = 3(3q^2 + 2q)$ ✓
- $n = 3q + 2$: $n^2 - 1 = 9q^2 + 12q + 3 = 3(3q^2 + 4q + 1)$ ✓ Both cases give a multiple of 3. ■

2.4 Let the integers be $2a+1$ and $2b+1$: their product is $4ab+2a+2b+1 = 2(2ab+a+b)+1$, which is one more than an even number – odd. ■

3.1 $m^2 + 3m + 2 = (m + 1)(m + 2)$. Cases: $m = 3q + 1$ or $m = 3q + 2$.

- $m = 3q + 1$: $(3q + 2)(3q + 3) = 3(3q + 2)(q + 1)$ – multiple of 3.
- $m = 3q + 2$: $(3q + 3)(3q + 4) = 3(q + 1)(3q + 4)$ – multiple of 3. Exhausting all non-multiples of 3, the expression is always a multiple of 3. ■ Check: $m = 1$: 6; $m = 2$: 12; $m = 4$: 30; $m = 5$: 42 – all divisible by 3 ✓.

3.2 A square's last digit depends only on its root's last digit. Exhaust the ten endings:

$$0^2 \to 0,\ 1^2 \to 1,\ 2^2 \to 4,\ 3^2 \to 9,\ 4^2 \to 6,\ 5^2 \to 5,\ 6^2 \to 6,\ 7^2 \to 9,\ 8^2 \to 4,\ 9^2 \to 1$$

The possible endings are $0, 1, 4, 5, 6, 9$ – never 8. ■

Local review guidance: complete case structure or a valid counterexample; algebra carried correctly; the conclusion states what was proved, with the case list justified. This is authored classroom guidance.