Exercise walk-through

7.2
The ellipse and the hyperbola

Pearson Edexcel · International A-Level · Pure Mathematics

The ellipse and the hyperbola

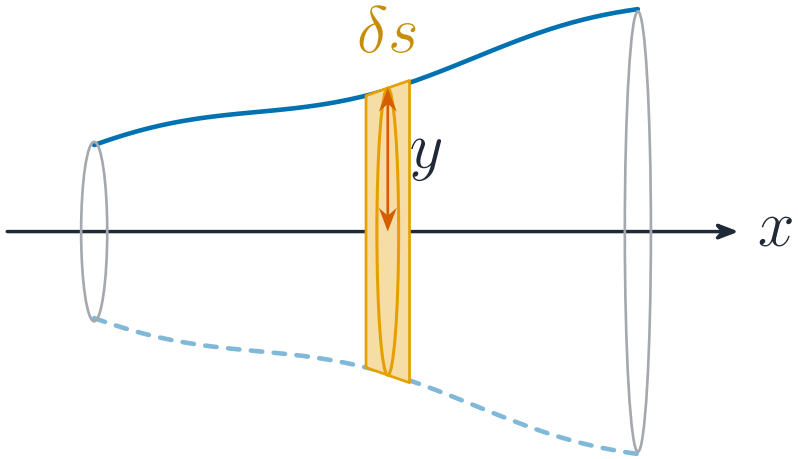
- For a point P on either curve, the distance from P to a **focus** equals e times the distance from P to the corresponding **directrix** : $PS = e \times PN$.
- The **ellipse** is a closed curve; the **hyperbola** has two branches.
- The number e is the **eccentricity** .
- **Worked example 4.** Find the **eccentricity** and foci of the **hyperbola** $\frac{x^2}{64} - \frac{y^2}{49} = 1$.

Integrating hyperbolic and inverse functions

- For a square root of a quadratic in the denominator, complete the square first.
- To integrate an inverse function, use integration by parts with $u =$ the inverse function and $\dfrac{dv}{dx} = 1$.
- **Worked example 7.** Find $\displaystyle\int \frac{1}{\sqrt{4x^2+8x+9}}\,dx$.
- Since $4x^2+8x+9 = 4\left((x+1)^2+\frac{5}{4}\right)$,

$$\int \frac{dx}{\sqrt{4x^2+8x+9}} = \frac{1}{2}\int \frac{dx}{\sqrt{(x+1)^2+\frac{5}{4}}} = \frac{1}{2}\operatorname{arsinh}\frac{2(x+1)}{\sqrt{5}} + c$$

Figure: Substitutions, reduction formulae and surfaces of revolution



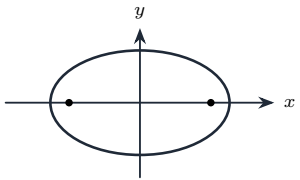
band area $\approx 2\pi y\,\delta s$

$$S = 2\pi \int |y| \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

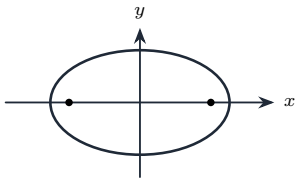
- A **surface of revolution** is made of thin bands, each with area about $2\pi y\,\delta s$.
- Read the axes and units first.

Exercises 2.1·2.2

2.1 For the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, find the eccentricity, the coordinates of the foci, and the equations of the directrices. [3]

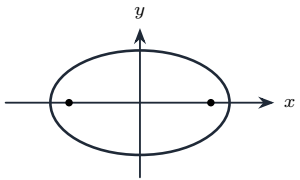


2.2 For the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, find the eccentricity, the foci and the directrices. [3]



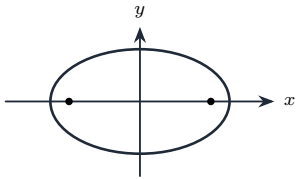
Exercises 2.1·2.2

2.1 For the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$, find the eccentricity, the coordinates of the foci, and the equations of the directrices. [3]



2.1 $9=25(1-e^2)$ gives $e=\frac{4}{5}$. Foci $(\pm ae,0)=(\pm 4,0)$; directrices $x=\pm\frac{a}{e}=\pm\frac{25}{4}$.

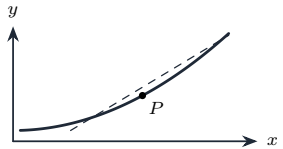
2.2 For the hyperbola $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$, find the eccentricity, the foci and the directrices. [3]



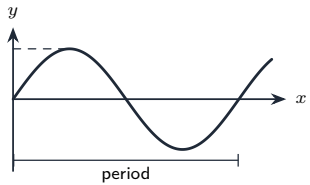
2.2 $16=9(e^2-1)$ gives $e^2=\frac{25}{9}$, $e=\frac{5}{3}$. Foci $(\pm 5,0)$; directrices $x=\pm\frac{9}{5}$.

Exercises 2.3·2.4

2.3 Find the equations of the tangents to $\frac{x^2}{4} + \frac{y^2}{3} = 1$ with gradient $\frac{1}{2}$. [4]

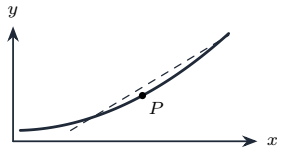


2.4 Show that the point $(4\sec\theta, 3\tan\theta)$ lies on $\frac{x^2}{16} - \frac{y^2}{9} = 1$ for all values of θ. [3]



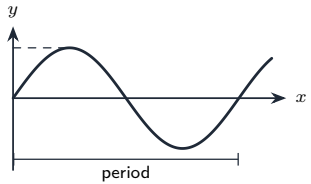
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2.3 With $m=\frac{1}{2}$: $c^2=a^2m^2+b^2=4\cdot\frac{1}{4}+3=4$, so $c=\pm2$ and the tangents are $y=\frac{1}{2}x\pm2$. (Substituting either into the ellipse gives a quadratic with zero discriminant.)

2.4 Show that the point $(4\sec\theta, 3\tan\theta)$ lies on $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$ for all values of θ. [3]

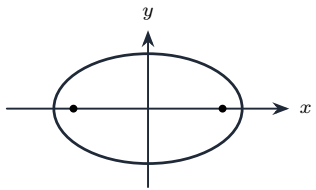


2.4 $\dfrac{(4\sec\theta)^2}{16}-\dfrac{(3\tan\theta)^2}{9}=\sec^2\theta-\tan^2\theta=1$ for all θ ✓.

Exercise 3.1

3.1 *(Edexcel IAL WFM03 June 2025 Q9a–c, condensed.)* The hyperbola H has equation $\frac{x^2}{64} - \frac{y^2}{49} = 1$ and eccentricity e.

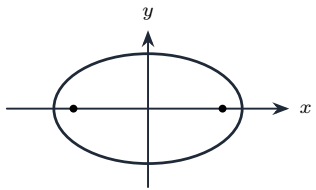
(a) Show that $e = \frac{\sqrt{113}}{8}$. (b) Use calculus to show that the tangent to H at $P(8\sec\theta, 7\tan\theta$ has equation $\frac{x}{8}\sec\theta - \frac{y}{7}\tan\theta = 1$. (c) Write down the coordinates of the point Q where this tangent meets the y-axis. [7]



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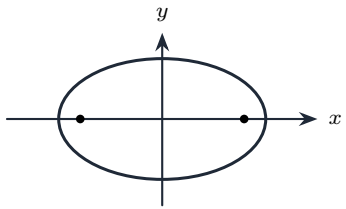


3.1 (a) $a^2 = 64$, $b^2 = 49$: $49 = 64(e^2 - 1)$, so $e^2 = \frac{113}{64}$ and $e = \frac{\sqrt{113}}{8}$. (b) $x = 8\sec\theta$, $y = 7\tan\theta$: $\dfrac{dy}{dx} = \dfrac{7\sec^2\theta}{8\sec\theta\tan\theta}$. The tangent: $y - 7\tan\theta = \dfrac{7\sec\theta}{8\tan\theta}(x - 8\sec\theta)$; multiplying through by $\frac{8\tan\theta}{7}$ and collecting, $\frac{x}{8}\sec\theta - \frac{y}{7}\tan\theta = \sec^2\theta - \tan^2\theta = 1$. (c) At $x = 0$: $-\frac{y}{7}\tan\theta = 1$, so $Q = (0, -7\cot\theta)$.

Exercise 3.2

3.2 *(Original.)* The hyperbola K has equation $\dfrac{x^2}{49} - \dfrac{y^2}{32} = 1$.

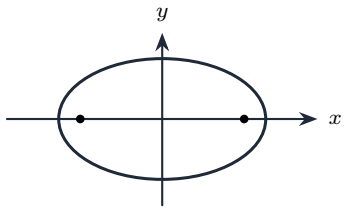
(a) Find its eccentricity and the coordinates of its foci. (b) Verify that the point $(7\sec\theta, 4\sqrt{2}\tan\theta)$ lies on K for all θ. [5]



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3.2 *(Original.)* The hyperbola K has equation $\dfrac{x^2}{49} - \dfrac{y^2}{32} = 1$.

(a) Find its eccentricity and the coordinates of its foci. (b) Verify that the point $(7\sec\theta, 4\sqrt{2}\tan\theta)$ lies on K for all θ. [5]



3.2 (a) $a^2 = 49$, $b^2 = 32$: $32 = 49(e^2 - 1)$, so $e^2 = \frac{81}{49}$, $e = \frac{9}{7}$; foci $(\pm ae, 0) = (\pm 9, 0)$. (b) $\dfrac{(7\sec\theta)^2}{49} = \sec^2\theta$ and $\dfrac{(4\sqrt{2}\tan\theta)^2}{32} = \frac{32\tan^2\theta}{32} = \tan^2\theta$, so the left side is $\sec^2\theta - \tan^2\theta = 1$ ✓.

Local review guidance: correct b^2–e relation for the curve type (sign!); foci and directrices placed with ae and a/e; tangents from $\frac{dy}{dx}$ at the parametric point, or the $c^2 = a^2m^2 \pm b^2$ condition for given gradients. This is authored classroom guidance.

Exercise 3.3 (a)

3.3 The ellipse E has equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$. The point $P(5\cos\theta,\ 4\sin\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on E. The tangent to E at P meets the x-axis at A and the y-axis at B.

(a) Show that an equation of the tangent at P is $4x\cos\theta + 5y\sin\theta = 20$. [4]

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(a)

- parametric gradient, $\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}$:

$$\frac{dy}{dx}=\frac{4\cos\theta}{-5\sin\theta}$$

- $y-y_1=m(x-x_1)$, then multiply by $5\sin\theta$:

$$5y\sin\theta-20\sin^2\theta=-4x\cos\theta+20\cos^2\theta\ \Rightarrow\ 4x\cos\theta+5y\sin\theta=20(\cos^2\theta+\sin^2\theta)=20$$

Exercise 3.3 (b)

3.3 The ellipse E has equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$. The point $P(5\cos\theta,\ 4\sin\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on E. The tangent to E at P meets the x-axis at A and the y-axis at B.

(b) Find the coordinates of A and B. [2]

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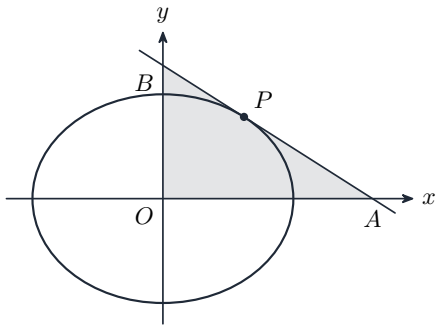
(b)

- $y = 0$: $A\left(\dfrac{5}{\cos\theta},\ 0\right)$; $x = 0$: $B\left(0,\ \dfrac{4}{\sin\theta}\right)$

Exercise 3.3 (c)

3.3 The ellipse E has equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The point $P(5\cos\theta,\ 4\sin\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on E. The tangent to E at P meets the x-axis at A and the y-axis at B.

(c) Show that the area of triangle OAB is $\frac{20}{\sin 2\theta}$, and find the least possible area. [3]

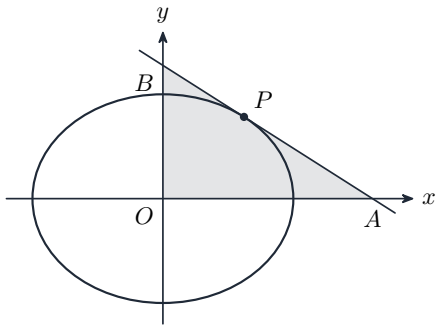


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Exercise 3.3 (c)

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(c)

- right-angled triangle, area $= \frac{1}{2} \times OA \times OB$, with $2\sin\theta\cos\theta = \sin 2\theta$:

$$\text{area} = \tfrac{1}{2} \cdot \frac{5}{\cos\theta} \cdot \frac{4}{\sin\theta} = \frac{10}{\sin\theta\cos\theta} = \frac{20}{\sin 2\theta}$$

- least when $\sin 2\theta = 1$, at $\theta = \frac{\pi}{4}$: least area $=$ **20**

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