Exercise walk-through

6.6
Maclaurin–Taylor series and polar coordinates

Pearson Edexcel · International A-Level · Pure Mathematics

The method of differences

- The **method of differences** sums a series by writing each term as a difference, usually using **partial fractions**, so that most of the terms cancel.
- Write out the terms for $r = 1$, $r = 2$, ..., $r = n - 1$ and $r = n$ to show clearly what cancels; this earns marks.
- **Worked example 3.** Show that $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$.
- Check with $n = 1$: both sides are $\frac{1}{6}$.

$$\frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}$$

Polar coordinates and areas

- In **polar coordinates** , a point is described by (r, θ): its distance r from the pole (the origin), and the angle θ measured anticlockwise from the **initial line** (the positive x-axis).
- To convert, use $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$.
- **Worked example 11.** Convert $r = 4\cos\theta$ to Cartesian form.
- Multiply by r: $r^2 = 4r\cos\theta$, so $x^2 + y^2 = 4x$, which is the circle $(x-2)^2 + y^2 = 4$.

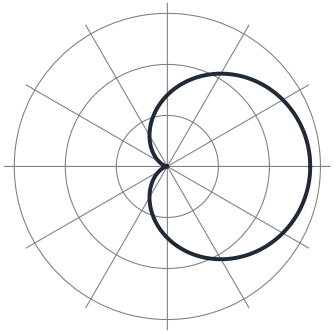
$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Key terms (2)

- **Integrating factor**: $e^{\int P\,dx}$, which turns the left side of $y' + Py = Q$ into the derivative of a product.
- **Complementary function**: the general solution of the equation with the right side set to zero.
- **Particular integral**: one solution of the complete differential equation.
- **Polar coordinates**: a distance from the pole and an angle from the **initial line**.

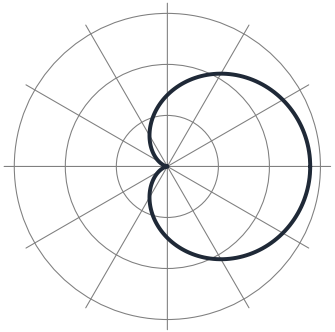
Exercise 2.1

2.1 Find the Maclaurin series of $\ln(1+2x)$ up to and including the term in x^3, stating the range of validity. [3]



Exercise 2.1

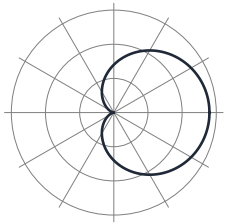
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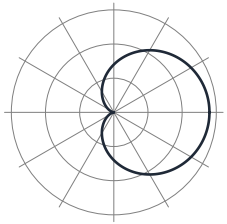
2.1 $f' = \frac{2}{1+2x}$, $f'' = \frac{-4}{(1+2x)^2}$, $f''' = \frac{16}{(1+2x)^3}$; at 0: $0, 2, -4, 16$. So $\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 - \cdots$, valid for $|2x| < 1$, i.e. $-\frac{1}{2} < x < \frac{1}{2}$.

Exercises 2.2·2.3

2.2 Find the first three non-zero terms of the Maclaurin series of $\sin x \cos x$. [2]

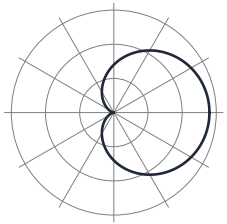


2.3 Convert $r = 4\cos\theta$ to a Cartesian equation and describe the curve. [3]



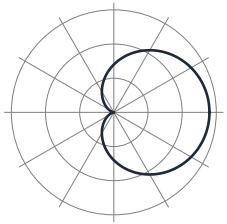
Exercises 2.2·2.3

2.2 Find the first three non-zero terms of the Maclaurin series of $\sin x \cos x$. [2]



2.2 $\sin x \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\left[2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{120} - \cdots\right] = x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \cdots$

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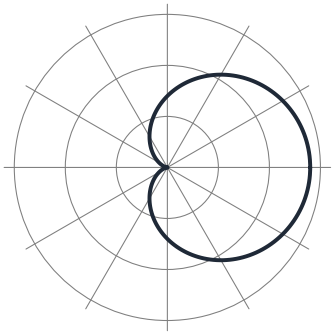


2.3 Multiply by r: $r^2 = 4r\cos\theta$, so $x^2 + y^2 = 4x$, i.e. $(x-2)^2 + y^2 = 4$: a circle, centre $(2, 0)$, radius 2, through the pole.

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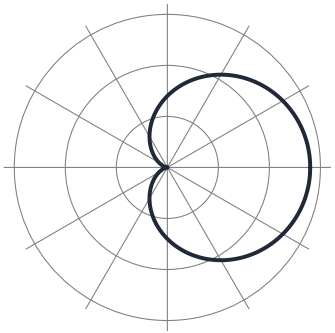
Exercise 2.4

2.4 Find the exact area enclosed by the cardioid $r = 2(1 + \cos\theta)$. [3]



Exercise 2.4

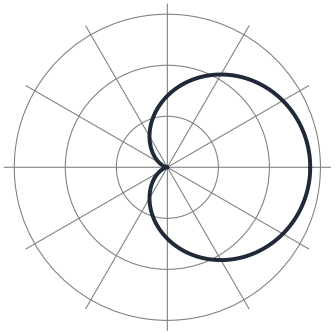
2.4 Find the exact area enclosed by the cardioid $r = 2(1 + \cos\theta)$. [3]



2.4 $\frac{1}{2}\int_{-\pi}^{\pi} 4(1+\cos\theta)^2\,d\theta = 2\int_{-\pi}^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta = 2(2\pi+0+\pi) = 6\pi$ (since $\int_{-\pi}^{\pi}\cos^2\theta\,d\theta = \pi$).

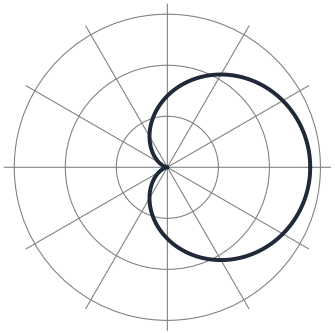
Exercise 3.1

3.1 *(Original.)* Given that $\dfrac{dy}{dx} = x^2 + y$ and $y = 1$ at $x = 0$, find the Taylor series for y about $x = 0$ up to and including the term in x^3. [6]



Exercise 3.1

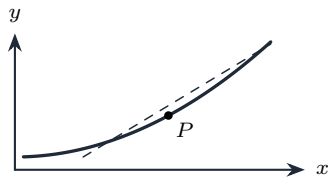
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3.1 $y'(0) = 0 + 1 = 1$. Differentiating the equation: $y'' = 2x + y'$, so $y''(0) = 0 + 1 = 1$. Again: $y''' = 2 + y''$, so $y'''(0) = 3$. The series is $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2}$. (Check at the centre: the equation gives $y''(0) = 2(0) + y'(0) = 1$, matching the series' quadratic term $\frac{1}{2}y''(0) = \frac{1}{2}$ ✓.)

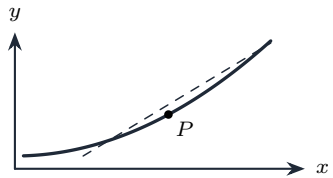
Exercise 3.2

3.2 *(Edexcel IAL WFM02 June 2025 Q2.)* Given that y satisfies $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = e^{2x/9}$ and that $y = 2$ and $\frac{dy}{dx} = -1$ at $x = 3$, find the Taylor series for y about $x = 3$, up to and including the term in $(x-3)^3$. [8]



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3.2 *(Edexcel IAL WFM02 June 2025 Q2.)* Given that y satisfies $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = e^{2x/9}$ and that $y = 2$ and $\dfrac{dy}{dx} = -1$ at $x = 3$, find the Taylor series for y about $x = 3$, up to and including the term in $(x-3)^3$. [8]



3.2 At $x = 3$: $y'' = 4y' + 5y + e^{2/3} = -4 + 10 + e^{2/3} = 6 + e^{2/3}$. Differentiating the equation: $y''' = 4y'' + 5y' + \frac{2}{9}e^{2x/9}$, so $y'''(3) = 4(6 + e^{2/3}) - 5 + \frac{2}{9}e^{2/3} = 19 + \frac{38}{9}e^{2/3}$. Therefore $y = 2 - (x-3) + \frac{6+e^{2/3}}{2}(x-3)^2 + \frac{19+\frac{38}{9}e^{2/3}}{6}(x-3)^3$, i.e. $y = 2 - (x-3) + \left(3 + \frac{1}{2}e^{2/3}\right)(x-3)^2 + \left(\frac{19}{6} + \frac{19}{27}e^{2/3}\right)(x-3)^3$.

Local review guidance: derivatives obtained by differentiating the equation, not by guessing the function; factorial divisions correct term by term; polar areas use $\frac{1}{2}r^2$ inside the integral with exact trigonometric integrals. This is authored classroom guidance.

Exercise 3.3 (a)

3.3 The figure shows the curve C with polar equation $r = 3 + 2\cos\theta$, $0 \leq \theta < 2\pi$. The region R, shaded, is bounded by C, the initial line and the half-line $\theta = \frac{\pi}{2}$.

(a) Find the exact area of R. [5]

Exercise 3.3 (a)

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(a) Find the exact area of R. [5]

(a)

- area in polar coordinates, $A = \frac{1}{2}\int r^2\,d\theta$, with $r^2 = 9 + 12\cos\theta + 4\cos^2\theta$
- use $4\cos^2\theta = 2 + 2\cos 2\theta$, so $r^2 = 11 + 12\cos\theta + 2\cos 2\theta$:

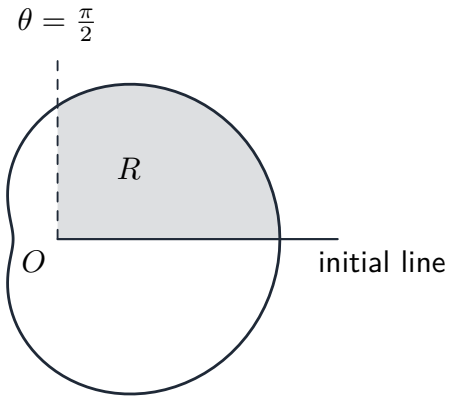
$$A = \tfrac{1}{2}\int_0^{\frac{\pi}{2}} (11 + 12\cos\theta + 2\cos 2\theta)\,d\theta = \tfrac{1}{2}\Big[11\theta + 12\sin\theta + \sin 2\theta\Big]_0^{\frac{\pi}{2}}$$

$$A = \tfrac{1}{2}\left(\tfrac{11\pi}{2} + 12 + 0\right) = \mathbf{\tfrac{11\pi}{4} + 6}$$

Exercise 3.3 (b)

3.3 The figure shows the curve C with polar equation $r = 3 + 2\cos\theta$, $0 \leq \theta < 2\pi$. The region R, shaded, is bounded by C, the initial line and the half-line $\theta = \frac{\pi}{2}$.

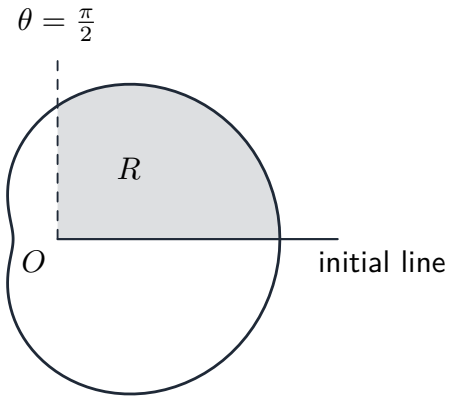
(b) Show that the tangent to C is parallel to the initial line where $4\cos^2\theta + 3\cos\theta - 2 = 0$, and find the value of r at the point where this happens with $0 < \theta < \frac{\pi}{2}$. [4]



Exercise 3.3 (b)

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Full solution on next slide

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Full solution 3.3 (b)

(b)

- parallel to the initial line where $\frac{dy}{d\theta} = 0$, with $y = r\sin\theta = 3\sin\theta + 2\sin\theta\cos\theta = 3\sin\theta + \sin 2\theta$:

$$\frac{dy}{d\theta} = 3\cos\theta + 2\cos 2\theta = 3\cos\theta + 2(2\cos^2\theta - 1) = 0 \;\Rightarrow\; 4\cos^2\theta + 3\cos\theta - 2 = 0$$

- quadratic formula, taking the root with $\cos\theta > 0$: $\cos\theta = \frac{-3+\sqrt{41}}{8} = 0.4254$
- $r = 3 + 2\cos\theta = 3 + \frac{-3+\sqrt{41}}{4} = \frac{\mathbf{9}+\sqrt{\mathbf{41}}}{\mathbf{4}} = 3.85$