Exercise walk-through

6.4
Loci, regions and $z \rightarrow w$ transformations

Pearson Edexcel · International A-Level · Pure Mathematics

Further Pure Mathematics 2: inequalities, further complex numbers and differential equations

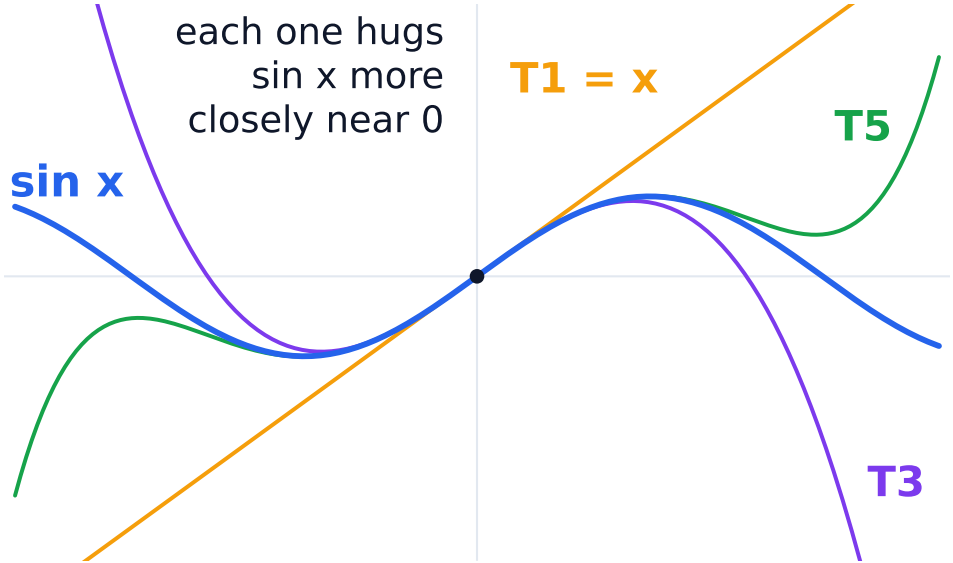
- solve inequalities involving the modulus sign, and sum series by the **method of differences**;
- use **Euler's relation** and **de Moivre's theorem**, and find the nth roots of complex numbers;
- sketch loci in the Argand diagram, and find images under transformations $w = \frac{az+b}{cz+d}$;

The method of differences

- The **method of differences** sums a series by writing each term as a difference, usually using **partial fractions**, so that most of the terms cancel.
- Write out the terms for $r = 1$, $r = 2$, ..., $r = n - 1$ and $r = n$ to show clearly what cancels; this earns marks.
- **Worked example 3.** Show that $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$.
- Check with $n = 1$: both sides are $\frac{1}{6}$.

$$\frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}$$

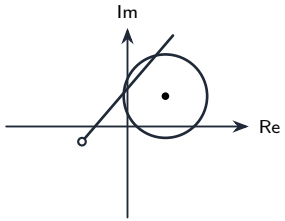
Figure: Maclaurin and Taylor series



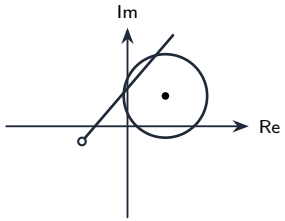
- The curves compare $\sin x$ with $T_1 = x$, $T_3 = x - x^3/6$ and $T_5 = x - x^3/6 + x^5/120$.
- Read the axes and units first.

Exercises 2.1·2.2

2.1 Describe fully the locus given by $|z - 2i| = 3$. [2]

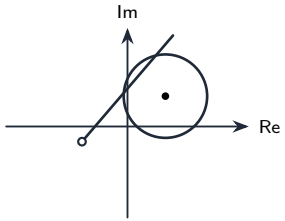


2.2 Describe fully the locus given by $|z - 3| = |z + 1|$. [2]



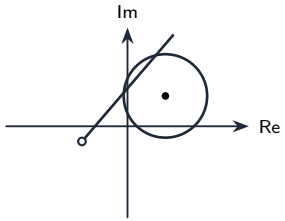
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2.1 Circle, centre $2i$ (the point $(0, 2)$), radius 3.

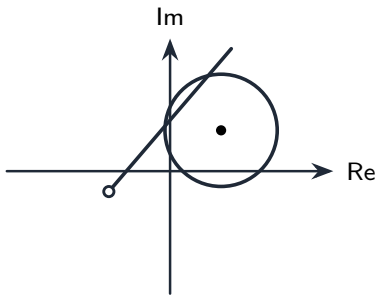
2.2 Describe fully the locus given by $|z - 3| = |z + 1|$. [2]



2.2 Perpendicular bisector of the points 3 and -1: the vertical line $\operatorname{Re} z = 1$.

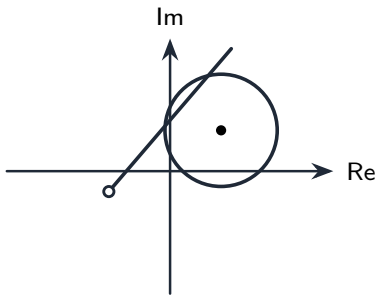
Exercise 2.3

2.3 Show that the locus $|z - 2| = 2|z + 1|$ is a circle, and state its centre and radius. [4]



Exercise 2.3

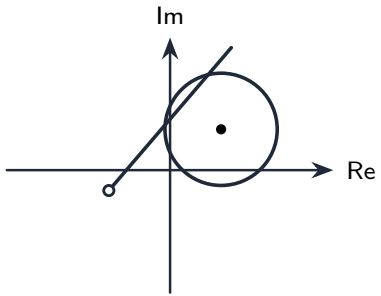
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2.3 With $z=x+iy$: $(x-2)^2+y^2=4\left[(x+1)^2+y^2\right]$, so $x^2-4x+4+y^2=4x^2+8x+4+4y^2$, i.e. $3x^2+12x+3y^2=0$, so $x^2+4x+y^2=0$, i.e. $(x+2)^2+y^2=4$: circle, centre $(-2,0)$, radius 2. (Check: $z=0$ gives $|-2|=2|1|$ ✓ and 0 lies on it.)

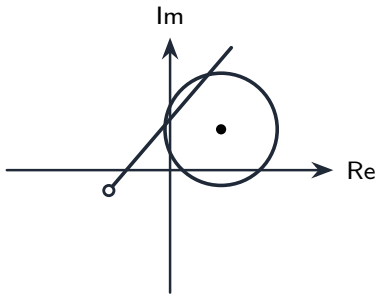
Exercise 2.4

2.4 Describe the locus given by $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$. [3]



Exercise 2.4

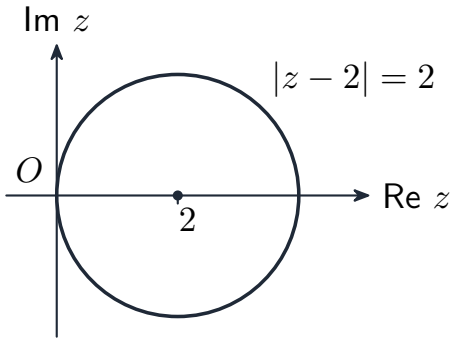
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2.4 $\arg\left(\frac{z-1}{z+1}\right) = \arg(z-1) - \arg(z+1)$ is the angle at z subtended by the segment from -1 to 1. Equal to $\frac{\pi}{2}$: the semicircle $x^2 + y^2 = 1$, $y > 0$, excluding the endpoints ± 1.

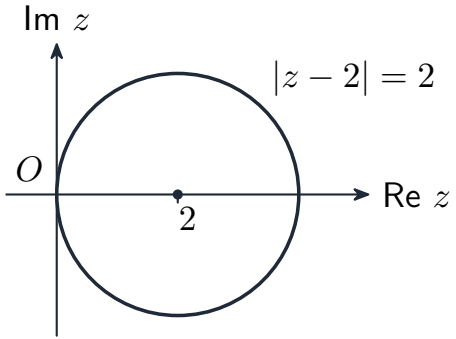
Exercise 3.1

3.1 *(Original.)* The transformation T from the z-plane to the w-plane is given by $w = \dfrac{1}{z}$, $z \neq 0$. Show that T maps the circle $|z - 2| = 2$ onto the straight line $\operatorname{Re} w = \frac{1}{4}$, and sketch this line. [5]



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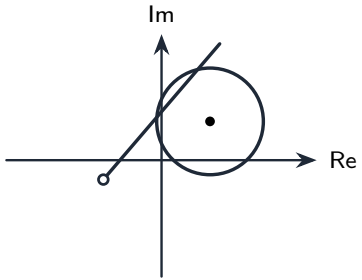
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3.1 $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$. Then $|\frac{1}{w} - 2| = 2$ gives $|1 - 2w| = 2|w|$. Squaring: $1 - 2w - 2\bar{w} + 4|w|^2 = 4|w|^2$, so $w + \bar{w} = \frac{1}{2}$, i.e. $\operatorname{Re} w = \frac{1}{4}$ – a vertical line in the w-plane, $\frac{1}{4}$ unit to the right of the imaginary axis. (Check: $z = 4$ is on the circle, and $z = 4 \mapsto w = \frac{1}{4}$ ✓.)

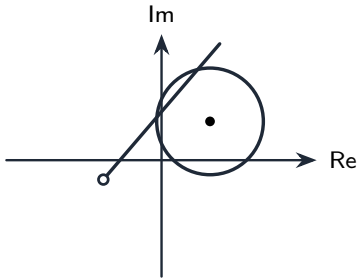
Exercise 3.2

3.2 *(Edexcel IAL WFM02 June 2025 Q3.)* The transformation T from the z-plane to the w-plane is given by $w = \dfrac{9i - iz}{1 + z}$, $z \neq -1$. The point P represents the complex number z, where $|z| = 1$. (a) Sketch the locus of P as z varies. (b) Given that the image under T of the locus of P, where $z \neq -1$, is the line l, determine, in simplest form, a Cartesian equation for l. (c) Write down the image under T of the point $z = i$ and verify that it lies on l. [9]



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Full solution on next slide

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Full solution 3.2

3.2 (a) Unit circle, centre the origin. (b) $z = \dfrac{9i - w}{w + i}$, so $|9i - w| = |w + i| \cdot |z| = |w + i|$. With $w = u + iv$: $9i - w = -u + i(9 - v)$, so $u^2 + (9 - v)^2 = u^2 + (v + 1)^2$, giving $81 - 18v = 2v + 1$, i.e. $20v = 80$: l is $v = 4$. (c) $z = i$: $w = \dfrac{9i - i^2}{1 + i} = \dfrac{1 + 9i}{1 + i} = \dfrac{(1 + 9i)(1 - i)}{2} = \dfrac{10 + 8i}{2} = 5 + 4i$, whose imaginary part is 4, so it lies on l ✓.

Local review guidance: z made the subject before any modulus work; modulus equation expanded in u, v; line/circle description with geometry named (bisector, centre, radius). This is authored classroom guidance.

Exercise 3.3 (a)

3.3 The Argand diagram shows a region R, shaded. Its boundary consists of part of a circle and part of a straight line.

(a) Write down two inequalities, in terms of z, that together define R. [3]

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(a)

- the circle has centre $2 + 2i$ and radius 2, and R is inside it: $|z - 2 - 2i| \leq 2$
- the line $\mathrm{Re}(z) = 2$ is the perpendicular bisector of 0 and 4, and R is on the side of O: $|z| \leq |z - 4|$, or $\mathrm{Re}(z) \leq 2$

Exercise 3.3 (b)

3.3 The Argand diagram shows a region R, shaded. Its boundary consists of part of a circle and part of a straight line.

(b) Find the greatest value of $|z|$ for z in R, and the value of z at which it occurs. [3]

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(b) Find the greatest value of $|z|$ for z in R, and the value of z at which it occurs. [3]

(b)

- on the semicircle $z = (2 + 2\cos t) + (2 + 2\sin t)i$ with $\cos t \leq 0$:

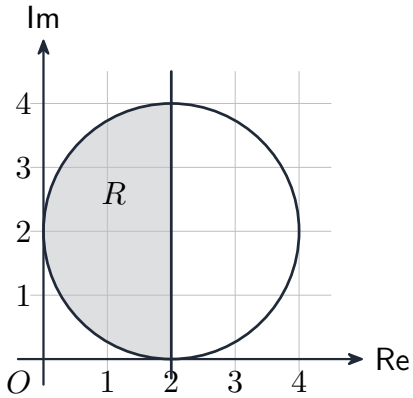
$$|z|^2 = (2 + 2\cos t)^2 + (2 + 2\sin t)^2 = 12 + 8(\cos t + \sin t)$$

- with $\cos t \leq 0$, $\cos t + \sin t$ is greatest at $t = \frac{\pi}{2}$, where it equals 1 (the farthest point of the full circle, in the direction of the centre, has $\operatorname{Re} z > 2$ and lies outside R)
- greatest $|z| = \sqrt{20} = \mathbf{2\sqrt{5}}$, at $z = \mathbf{2 + 4i}$

Exercise 3.3 (c)

3.3 The Argand diagram shows a region R, shaded. Its boundary consists of part of a circle and part of a straight line.

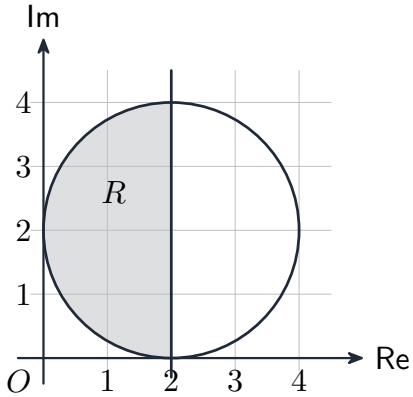
(c) Find the exact area of R. [1]



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(c)

- half of a circle of radius 2: $\frac{1}{2}\pi(2)^2 = \mathbf{2}\pi$