Exercise walk-through

6.3
Euler's relation, de Moivre's theorem and complex roots

Pearson Edexcel · International A-Level · Pure Mathematics

Further Pure Mathematics 2: inequalities, further complex numbers and differential equations

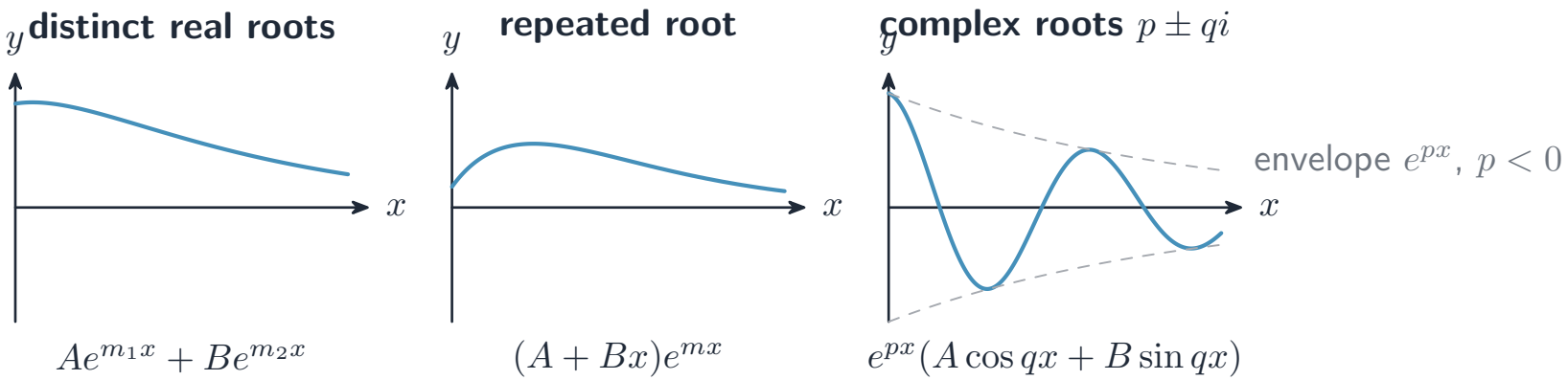
- solve inequalities involving the modulus sign, and sum series by the **method of differences**;
- use **Euler's relation** and **de Moivre's theorem**, and find the nth roots of complex numbers;
- sketch loci in the Argand diagram, and find images under transformations $w = \frac{az+b}{cz+d}$;

Euler's relation and de Moivre's theorem

- **Euler's relation** is
- So a complex number with modulus r and argument θ can be written $z = re^{i\theta}$, and then $z^n = r^n e^{in\theta}$.
- It also gives $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.
- **De Moivre's theorem** states that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

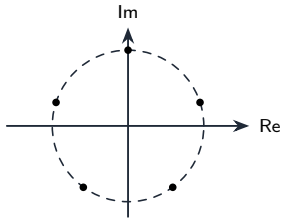
Figure: Second order differential equations



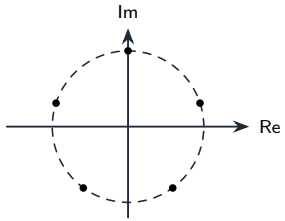
- The roots of the **auxiliary equation** set the shape of the **complementary function**; here every root has a negative real part.
- Match each part of the diagram to the text.

Exercises 2.1·2.2

2.1 Use de Moivre's theorem to express $\cos 3\theta$ as a polynomial in $\cos\theta$. [3]

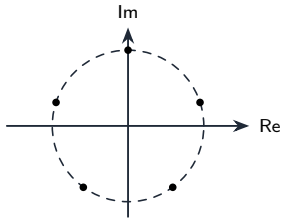


2.2 Express $\sin^3\theta$ in the form $a\sin\theta + b\sin 3\theta$. [3]



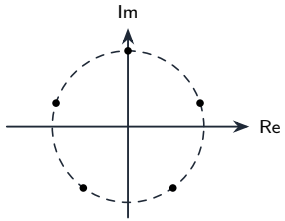
Exercises 2.1·2.2

2.1 Use de Moivre's theorem to express $\cos 3\theta$ as a polynomial in $\cos\theta$. [3]



2.1 Real part of $(c+is)^3$: $c^3 - 3cs^2 = c^3 - 3c(1-c^2) = 4\cos^3\theta - 3\cos\theta$.

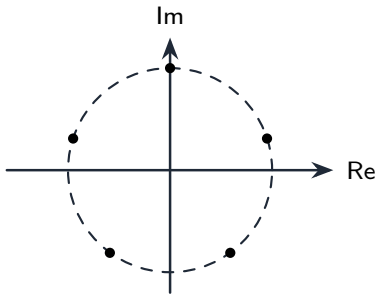
2.2 Express $\sin^3\theta$ in the form $a\sin\theta + b\sin 3\theta$. [3]



2.2 From $\sin 3\theta = 3s - 4s^3$ (imaginary part of $(c+is)^3$): $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$.

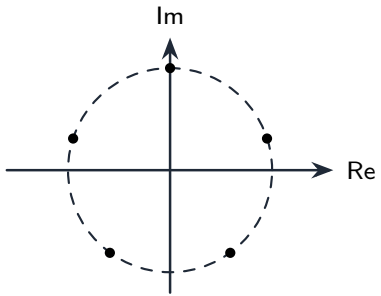
Exercise 2.3

2.3 Solve $z^3 = 8i$, giving the roots in the form $re^{i\theta}$ with exact θ. [4]



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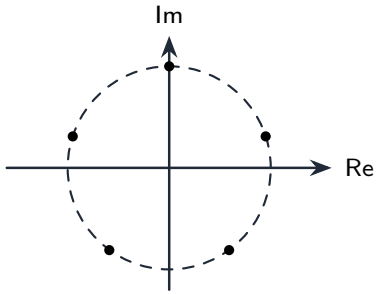
2.3 Solve $z^3 = 8i$, giving the roots in the form $re^{i\theta}$ with exact θ. [4]



2.3 $8i = 8e^{i\pi/2}$, so $z = 2e^{i(\pi/6+2\pi k/3)}$, $k = 0, 1, 2$: $2e^{i\pi/6}$, $2e^{i5\pi/6}$, $2e^{i3\pi/2}$. (Check: the product of the roots is $8e^{i(\pi/6+5\pi/6+3\pi/2)} = 8e^{i5\pi/2} = 8i$ ✓, matching $z_1z_2z_3 = -(-8i)$.)

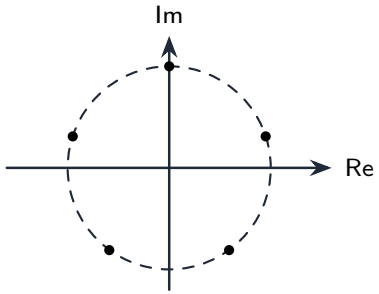
Exercise 3.1

3.1 *(Original.)* (a) Show that $\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$. (b) Hence evaluate $\int_0^{\pi/2} \cos^4\theta \, d\theta$. [7]



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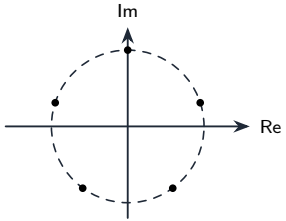
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3.1 (a) $\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$. (Or: real part of $(\cos\theta + i\sin\theta)^4$ is $c^4 - 6c^2s^2 + s^4$; substitute $c^2 = \frac{1+\cos 2\theta}{2}$, $s^2 = \frac{1-\cos 2\theta}{2}$.) (b) $\displaystyle\int_0^{\pi/2} \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)\, d\theta = \frac{1}{8}\left[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{8}\cdot\frac{3\pi}{2} = \frac{3\pi}{16}$.

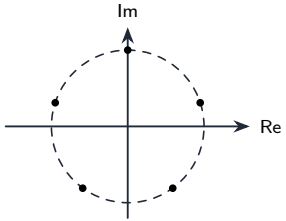
Exercise 3.2

3.2 *(Edexcel IAL WFM02 June 2025 Q1.)* (a) Express $-2\sqrt{2}-2\sqrt{6}i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. (b) Hence solve $z^5 = -2\sqrt{2} - 2\sqrt{6}i$, giving your answers in the form $re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$. [8]



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3.2 (a) Modulus $\sqrt{(-2\sqrt{2})^2+(-2\sqrt{6})^2}=\sqrt{32}=4\sqrt{2}$. Both components negative, and $\tan\theta=\frac{\sqrt{6}}{\sqrt{2}}=\sqrt{3}$ gives a reference angle $\frac{\pi}{3}$, so $\theta=-\pi+\frac{\pi}{3}=-\frac{2\pi}{3}$: $4\sqrt{2}\left(\cos\frac{2\pi}{3}\cdot(-1)\ldots\right)$, i.e. $4\sqrt{2}(\cos(-\frac{2\pi}{3})+i\sin(-\frac{2\pi}{3}))$. (b) $z=(4\sqrt{2})^{1/5}e^{i\left(\frac{-2\pi/3+2\pi k}{5}\right)}=\sqrt{2}\,e^{i\left(-\frac{2\pi}{15}+\frac{2\pi k}{5}\right)}$, $k=0,1,2,3,4$, i.e. arguments $-\frac{2\pi}{15}$, $\frac{4\pi}{15}$, $\frac{10\pi}{15}=\frac{2\pi}{3}$, $\frac{16\pi}{15}-2\pi=-\frac{14\pi}{15}$, $\frac{22\pi}{15}-2\pi=-\frac{8\pi}{15}$ (all in $-\pi<\theta\leq\pi$): $z\in\left\{\sqrt{2}e^{-2\pi i/15},\sqrt{2}e^{4\pi i/15},\sqrt{2}e^{2\pi i/3},\sqrt{2}e^{-14\pi i/15},\sqrt{2}e^{-8\pi i/15}\right\}$.

Local review guidance: modulus and argument found exactly with the correct quadrant; roots carry the $\frac{2\pi k}{n}$ spread and land inside the stated argument range; power identities simplified with $c^2+s^2=1$ substitutions. This is authored classroom guidance.