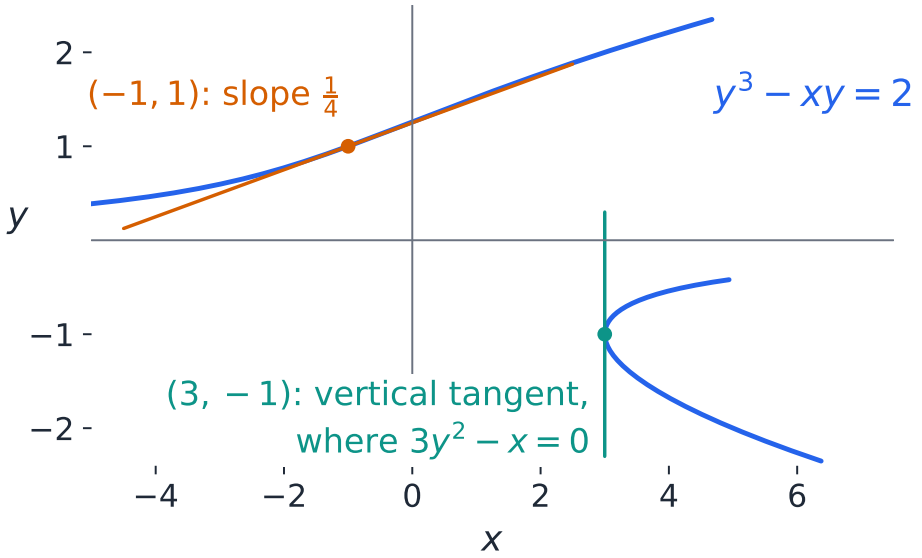
Exercise walk-through

4.4
Implicit differentiation and differential equations

Pearson Edexcel · International A-Level · Pure Mathematics

Figure: Implicit and parametric differentiation



- Where the denominator of $\frac{dy}{dx}$ is zero (and the numerator is not), the tangent is vertical
- Read the axes and units first.

Integration: substitution, parts and partial fractions

- **Substitution** (usually given): rewrite the integral in u, adjust dx via $\frac{du}{dx}$, integrate, substitute back. It is the chain rule reversed.
- **Integration by parts** : $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ — the product rule reversed. Know $\int \ln x\,dx = x\ln x - x + c$ and repeated parts for $\int x^2 e^x dx$-type integrals.

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

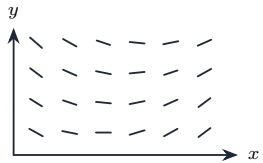
Integration: substitution, parts and partial fractions (2)

- **Partial fractions** turn a **rational** integrand into sums of $\frac{A}{ax+b}$ (integrating to logs) and $\frac{C}{(ax+b)^2}$ (to powers).
- Parts with $u = \mathrm{e}^x$, $\frac{dv}{dx} = \cos x$:
- Parts again on the new integral ($u = \mathrm{e}^x$, $\frac{dv}{dx} = \sin x$):

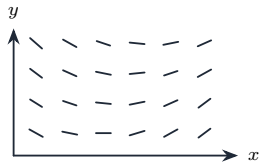
$$= \mathrm{e}^x \sin x + \mathrm{e}^x \cos x - \int \mathrm{e}^x \cos x \, dx$$

Exercises 2.1·2.2

2.1 Find $\dfrac{dy}{dx}$ for $x^2 + y^2 = 25$. [2]

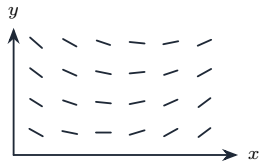


2.2 Find $\dfrac{dy}{dx}$ for $x^2 + 2xy = 3$. [3]



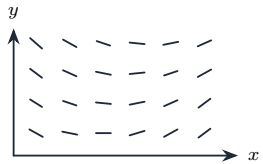
Exercises 2.1·2.2

2.1 Find $\dfrac{dy}{dx}$ for $x^2 + y^2 = 25$. [2]



2.1 $2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$.

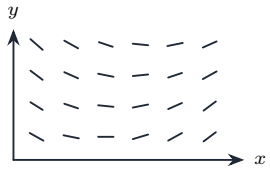
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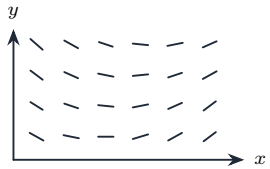
2.2 $2x + 2y + 2x\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x + y}{x}$.

Exercises 2.3·2.4

2.3 A cube's edge grows at 0.1 cm/s. Find $\dfrac{dV}{dt}$ when the edge is 5 cm. [3]

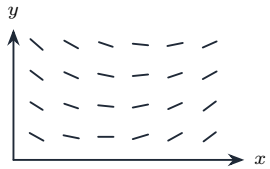


2.4 Solve $\dfrac{dy}{dx} = \dfrac{2x}{y}$ with $y = 3$ at $x = 0$. [4]



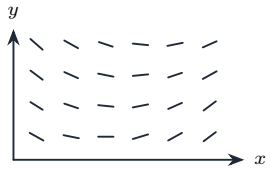
Exercises 2.3·2.4

2.3 A cube's edge grows at 0.1 cm/s. Find $\dfrac{dV}{dt}$ when the edge is 5 cm. [3]



2.3 $V = x^3$: $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt} = 3(25)(0.1) = 7.5$ cm^3/s.

2.4 Solve $\dfrac{dy}{dx} = \dfrac{2x}{y}$ with $y = 3$ at $x = 0$. [4]



2.4 $\int y\,dy = \int 2x\,dx \Rightarrow \frac{y^2}{2} = x^2 + c$; $y(0) = 3$ gives $c = \frac{9}{2}$, so $y^2 = 2x^2 + 9$.

Exercise 3.1

3.1 *(Adapted from Edexcel IAL WMA14 June 2025 Q2.)* A sealed spherical balloon loses air at a constant 12 cm^3 per second. Show that the rate of decrease of its radius is inversely proportional to the square of the radius, and find that rate when $r = 3$ cm. [5]



Exercise 3.1

3.1 *(Adapted from Edexcel IAL WMA14 June 2025 Q2.)* A sealed spherical balloon loses air at a constant 12 cm^3 per second. Show that the rate of decrease of its radius is inversely proportional to the square of the radius, and find that rate when $r = 3$ cm. [5]



3.1 $V = \frac{4}{3}\pi r^3$, $\dfrac{dV}{dr} = 4\pi r^2$; chain rule with $\dfrac{dV}{dt} = -12$:

$$-12 = 4\pi r^2 \frac{dr}{dt} \implies \frac{dr}{dt} = -\frac{3}{\pi r^2}$$

The rate of decrease $\dfrac{3}{\pi r^2}$ is a constant multiple of $\dfrac{1}{r^2}$ – inversely proportional to r^2. ■

At $r = 3$:

$$\frac{3}{9\pi} = \frac{1}{3\pi} \approx 0.106 \text{ cm/s (3 s.f.)}$$

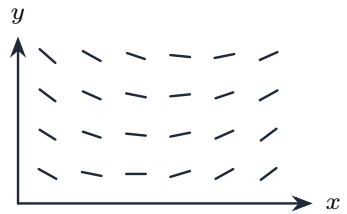
Check: units $\text{cm}^3/\text{s} \div \text{cm}^2 = \text{cm/s}$ ✓.

Exercise 3.2

3.2 *(Adapted from Edexcel IAL WMA14 June 2025 Q3.)* Solve the differential equation

$$y\frac{dy}{dx} = 6\sec 3x \tan 3x, \qquad y = 2 \text{ at } x = 0$$

giving your answer in the form $y^2 = g(x)$. [6]

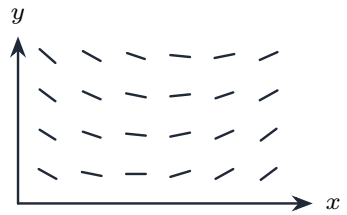


Exercise 3.2

3.2 *(Adapted from Edexcel IAL WMA14 June 2025 Q3.)* Solve the differential equation

$$y\frac{dy}{dx} = 6\sec 3x \tan 3x, \qquad y = 2 \text{ at } x = 0$$

giving your answer in the form $y^2 = g(x)$. [6]



3.2 Separate: $\int y\,dy = \int 6\sec 3x \tan 3x\,dx$ with $\int \sec 3x \tan 3x\,dx = \frac{1}{3}\sec 3x$:

$$\frac{y^2}{2} = 2\sec 3x + c \implies y^2 = 4\sec 3x + C$$

Full solution on next slide

Full solution 3.2

3.2 Separate: $\int y\,dy = \int 6\sec 3x\tan 3x\,dx$ with $\int \sec 3x\tan 3x\,dx = \frac{1}{3}\sec 3x$:

$$\frac{y^2}{2} = 2\sec 3x + c \implies y^2 = 4\sec 3x + C$$

Condition $y = 2$ at $x = 0$: $4 = 4(1) + C \Rightarrow C = 0$:

$$y^2 = 4\sec 3x$$

Check: $\sqrt{4} = 2$ at $x = 0$ ✓; differentiating back: $2y\dfrac{dy}{dx} = 12\sec 3x\tan 3x \Rightarrow y\dfrac{dy}{dx} = 6\sec 3x\tan 3x$ ✓.

Local review guidance: correct implicit/rate structure; exact derivative or particular solution; condition applied and the solution checked. This is authored classroom guidance.

Exercise 3.3 (a)

3.3 A tank is an inverted right circular cone of radius 3 m and height 6 m, with its vertex at the bottom. When the depth of water is h m, the volume of water is V m^3.

(a) Show that $V = \dfrac{\pi h^3}{12}$. [2]

Exercise 3.3 (a)

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(a) Show that $V = \dfrac{\pi h^3}{12}$. [2]

(a)

- similar triangles: $\dfrac{r}{h} = \dfrac{3}{6}$, so the water surface has radius $r = \frac{1}{2}h$
- volume of a cone, $V = \frac{1}{3}\pi r^2 h$:

$$V = \tfrac{1}{3}\pi \left(\tfrac{1}{2}h\right)^2 h = \frac{\pi h^3}{12}$$

Exercise 3.3 (b)

3.3 A tank is an inverted right circular cone of radius 3 m and height 6 m, with its vertex at the bottom. When the depth of water is h m, the volume of water is V m^3.

(b) Water leaks from a hole at the vertex at a rate modelled by $\dfrac{dV}{dt} = -k\sqrt{h}$, where t is the time in minutes and k is a positive constant. Show that $h^{\frac{3}{2}}\dfrac{dh}{dt} = -\dfrac{4k}{\pi}$. [2]

Exercise 3.3 (b)

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(b)

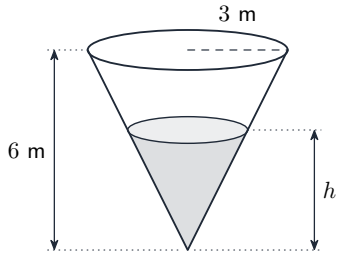
- chain rule, $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$, with $\frac{dV}{dh} = \frac{\pi h^2}{4}$:

$$\frac{\pi h^2}{4}\frac{dh}{dt} = -kh^{\frac{1}{2}} \Rightarrow h^{\frac{3}{2}}\frac{dh}{dt} = -\frac{4k}{\pi}$$

Exercise 3.3 (c)

3.3 A tank is an inverted right circular cone of radius 3 m and height 6 m, with its vertex at the bottom. When the depth of water is h m, the volume of water is V m^3.

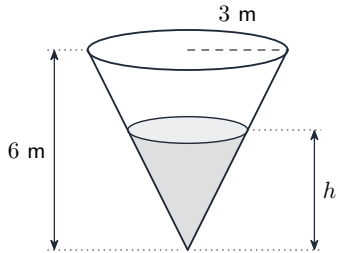
(c) When $t = 0$, $h = 4$, and when $t = 31$, $h = 1$. Solve the differential equation to find k in terms of π, and find the time at which the tank is empty. [5]



Exercise 3.3 (c)

3.3 A tank is an inverted right circular cone of radius 3 m and height 6 m, with its vertex at the bottom. When the depth of water is h m, the volume of water is V m^3.

(c) When $t = 0$, $h = 4$, and when $t = 31$, $h = 1$. Solve the differential equation to find k in terms of π, and find the time at which the tank is empty. [5]



(c)

- separate the variables and integrate:

$$\int h^{\frac{3}{2}}\,dh = \int -\frac{4k}{\pi}\,dt \;\Rightarrow\; \tfrac{2}{5}h^{\frac{5}{2}} = -\frac{4k}{\pi}t + c$$

- $t = 0$, $h = 4$: $c = \frac{2}{5}(32) = 12.8$; $t = 31$, $h = 1$: $\frac{2}{5} = -\frac{4k}{\pi}(31) + 12.8$, so $\frac{4k}{\pi} = 0.4$ and $k = \mathbf{0.1}\pi$
- empty when $h = 0$: $0 = -0.4t + 12.8$, so $t = \mathbf{32}$ minutes

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