5.5 Simple harmonic motion, phase and energy

Name: _____ Class: _____ Date: _____ **Total: 29 marks**

KEY WORDS Simple harmonic motion

Objective

Develop simple harmonic motion, phase and energy from the WPH15 specification and mapped 2025 exam demands. These are original training questions. Use the constants stated in each task.

1 Key knowledge and worked examples

■ The restoring condition

Simple harmonic motion (SHM) occurs when resultant acceleration is proportional to displacement from a fixed equilibrium position and directed towards it:

$$F = -k_s x \qquad a = -\omega^2 x \qquad \omega^2 = k_s/m$$

The negative sign is the restoring direction. Constant speed, repetition alone or a force merely pointing towards a centre does not establish SHM. For a vertically hanging spring, measure x from the loaded equilibrium position. Weight is already balanced there; the resultant for displacement is $-k_s x$.

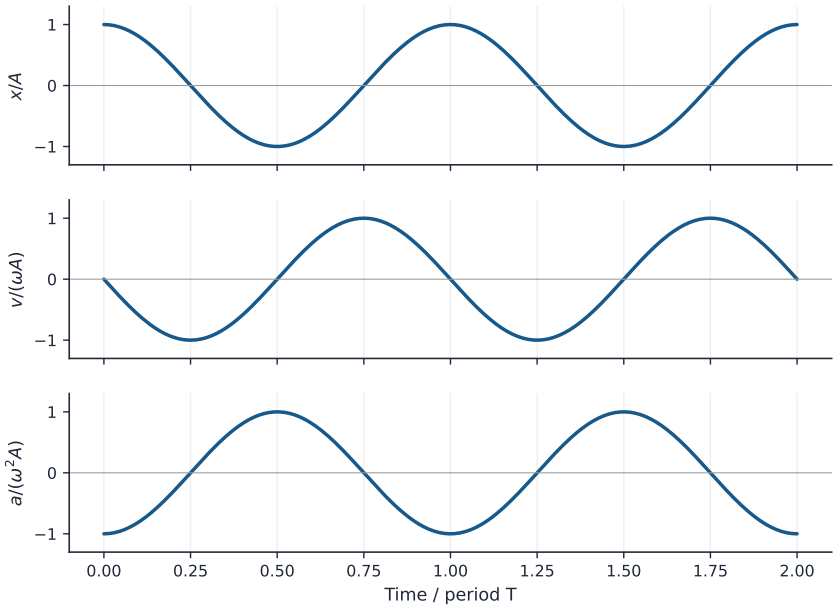
For release from positive maximum displacement at $t = 0$:

$$x = A_0 \cos \omega t \qquad v = -A_0 \omega \sin \omega t \qquad a = -A_0 \omega^2 \cos \omega t$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} \qquad v_{\max} = A_0 \omega \qquad |a|_{\max} = A_0 \omega^2$$

Here A_0 is amplitude, not radioactive activity. The phase depends on the chosen starting time; a sine displacement can describe the same motion with another origin. Differentiate graphically: the displacement–time gradient is velocity, and velocity–time gradient is acceleration. At maximum displacement, speed is zero and acceleration

points back towards equilibrium. At equilibrium, speed is greatest and acceleration is zero.



Worked example. A graph gives amplitude 0.030 m and period 0.50 s. Find maximum speed and acceleration.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.50\ \text{s}} = 12.6\ \text{rad}\,\text{s}^{-1}$$

$$v_{\max} = A_0\omega = 0.030\ \text{m} \times \frac{2\pi}{0.50\ \text{s}} = 0.377\ \text{m}\,\text{s}^{-1}$$

$$|a|_{\max} = A_0\omega^2 = 0.030\ \text{m} \times \left(\frac{2\pi}{0.50\ \text{s}}\right)^2 = 4.74\ \text{m}\,\text{s}^{-2}$$

Alternatively find maximum speed from a tangent to the displacement graph at equilibrium. Convert centimetres to metres. A negative straight a–x gradient is $-\omega^2$, so take its negative before the square root.

Periods and energy

For a mass on an ideal spring and for a simple pendulum at small angles:

$$T_{\text{spring}} = 2\pi\sqrt{m/k_s} \qquad T_{\text{pendulum}} = 2\pi\sqrt{l/g}$$

Pendulum length is pivot to bob centre. The small-angle period does not depend on bob mass. A period ratio for one unchanged spring gives $T_2/T_1 = \sqrt{m_2/m_1}$. Use total new mass before finding added mass.

Worked example. A 0.50 kg mass has period 0.80 s. An extra mass increases the period to 1.00 s on the same spring.

$$m_2 = m_1\left(\frac{T_2}{T_1}\right)^2 = 0.50\ \text{kg}\left(\frac{1.00\ \text{s}}{0.80\ \text{s}}\right)^2 = 0.781\ \text{kg}$$

$$m_{\text{added}} = m_2 - m_1 = (0.781 - 0.50)\ \text{kg} = 0.281\ \text{kg}$$

For an undamped spring oscillator, energy transfers between elastic potential and kinetic stores:

$$E = \frac{1}{2}k_s A_0^2 = \frac{1}{2}m\omega^2 A_0^2 \qquad E_p = \frac{1}{2}k_s x^2 \qquad E_k = \frac{1}{2}k_s(A_0^2 - x^2)$$

Thus the E_k–x graph is an inverted parabola: zero at $\pm A_0$, positive maximum at $x = 0$. The E_p–x graph opens upwards. Their sum is constant in the undamped model. In a damped system energy is transferred out, so amplitude decreases. Total energy including surroundings remains conserved.

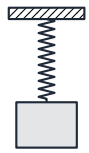
■ **CP16: infer an unknown mass from resonance**

Keep the same spring/support system. Attach several known masses, drive with small oscillations and vary frequency to find the largest steady amplitude for each mass. Record the resonant frequency, repeating slowly around each peak. With light damping it approximates natural frequency. Plot $1/f^2$ against total known mass; the spring model gives gradient $4\pi^2/k_s$. Use the calibration to infer an unknown mass from its resonant frequency.

Account for a hanger and, where significant, the effective moving mass of the spring. A nonzero intercept may represent this contribution; do not force the line through the origin without justification. Keep amplitudes small and below the elastic limit. Secure masses and stand, and keep feet clear. Measuring free periods can provide a useful independent comparison, but the specified practical uses resonant frequencies.

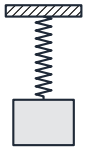
2 Practice

2.1 An oscillator has amplitude 0.040 m and period 0.80 s. Find angular frequency, maximum speed and maximum acceleration. [4]



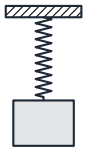
$A = 0.040$ m
$T = 0.80$ s

2.2 A 0.20 kg mass on a spring oscillates with period 0.60 s. With an extra mass on the same spring, period becomes 0.90 s. Find the extra mass. [3]



0.20 kg; 0.60 s
extra mass; 0.90 s

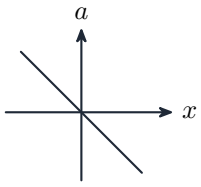
2.3 An undamped oscillator is released from $x = +A$ at $t = 0$. State x, velocity direction and acceleration direction at $T/4$, and x and velocity at $T/2$. Explain using the restoring condition. [4]



$x = +A$
released at $t = 0$

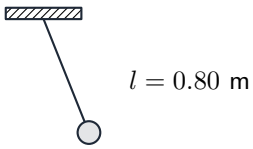
3 Exam-style questions

3.1 A straight acceleration–displacement graph has gradient $-25\ \text{s}^{-2}$. A mass 0.30 kg oscillates with amplitude 0.060 m. Find period, maximum kinetic energy and kinetic energy at $x = 0.020$ m. [6]

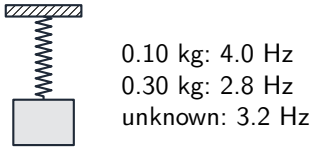


gradient $-25\ \text{s}^{-2}$
$A = 0.060$ m

3.2 A pendulum length is 0.80 m from pivot to bob centre. Calculate its small-angle period using $g = 9.81\ \mathrm{m\,s^{-2}}$. Explain the effect of doubling bob mass and identify two measurement choices improving a timing experiment. [5]



3.3 Known added masses 0.10 and 0.30 kg give resonant frequencies 4.0 and 2.8 Hz on one spring assembly. Assume light damping and a straight calibration of $1/f^2$ against added mass, retaining its intercept. Infer an unknown added mass giving 3.2 Hz. Explain how the intercept can arise and two controls for this calibration. [7]



Solutions

Original-task marking: M1 method, A1 accuracy, B1 reasoning.

2.1 Use period to obtain angular frequency.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80\ \text{s}} = 7.85\ \text{rad}\,\text{s}^{-1}$$

Angular frequency.

$$v_{max} = A\omega = (0.040\ \text{m})(7.854\ \text{s}^{-1}) = 0.314\ \text{m}\,\text{s}^{-1}$$

Speed.

$$|a|_{max} = A\omega^2 = (0.040\ \text{m})(7.854\ \text{s}^{-1})^2 = 2.47\ \text{m}\,\text{s}^{-2}$$

Acceleration.

2.2 Same spring gives squared period proportional to total mass.

$$m_2 = m_1(T_2/T_1)^2 = (0.20\ \text{kg})(0.90\ \text{s}/0.60\ \text{s})^2 = 0.45\ \text{kg}$$

Total mass.

$$m_{added} = m_2 - m_1 = (0.45 - 0.20)\ \text{kg} = 0.25\ \text{kg}$$

Added mass.

2.3 At $T/4$, x=0, velocity points in the negative x direction, acceleration is zero. At $T/2$, x=−A and v=0. Acceleration is always $-\omega^2 x$, towards equilibrium; the initial restoring acceleration produces the negative velocity.

3.1 The gradient is $-\omega^2$, so $\omega = 5.0\ \text{rad}\,\text{s}^{-1}$.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.0\ \text{s}^{-1}} = 1.26\ \text{s}$$

Period.

$$E_{k,max} = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}(0.30\ \text{kg})(25\ \text{s}^{-2})(0.060\ \text{m})^2 = 0.0135\ \text{J}$$

Energy method/value.

$$E_k = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}(0.30\ \text{kg})(25\ \text{s}^{-2})(0.060^2 - 0.020^2)\ \text{m}^2 = 0.0120\ \text{J}$$

Displaced energy.

3.2 Use the small-angle pendulum model.

$$T = 2\pi\sqrt{l/g} = 2\pi\sqrt{\frac{0.80\ \text{m}}{9.81\ \text{m}\,\text{s}^{-2}}} = 1.79\ \text{s}$$

Period. Doubling mass does not change the ideal period. Time many complete oscillations and divide by their number; repeat at small amplitude, measuring pivot-to-centre length.

3.3 Transform frequencies to reciprocal squares and interpolate that variable.

$$y_1 = f_1^{-2} = (4.0\ \text{Hz})^{-2} = 0.06250\ \text{s}^2$$

$$y_2 = f_2^{-2} = (2.8\ \text{Hz})^{-2} = 0.12755\ \text{s}^2$$
$$y_u = f_u^{-2} = (3.2\ \text{Hz})^{-2} = 0.097656\ \text{s}^2$$

Transformed values.

$$m_u = m_1 + (m_2 - m_1)\frac{y_u - y_1}{y_2 - y_1}$$
$$m_u = 0.10\ \text{kg} + (0.30 - 0.10)\ \text{kg}\frac{(0.097656 - 0.06250)\ \text{s}^2}{(0.127551 - 0.06250)\ \text{s}^2} = 0.208\ \text{kg}$$

Interpolation/value. Hanger/effective spring mass contributes a nonzero intercept. Keep the same spring and support with small amplitudes below elastic limit; vary frequency slowly around each steady peak and repeat the reading. Two points are provided for this task; a real calibration needs more to test linearity.