Mechanics 2

Edexcel IAL M2 (WME02/01); the applied unit after M1 in IAL Mathematics

Mechanics 2: projectiles, centres of mass, energy and collisions

- use calculus for motion with variable acceleration, and model projectiles;
- find centres of mass of particles, laminas and wire frameworks, and the angle at which a suspended body hangs;
- use work, energy and **power**;

Mechanics 2: projectiles, centres of mass, energy and collisions (2)

- solve direct collisions and successive impacts with restitution;
- solve equilibrium problems for rigid bodies.

Kinematics with variable acceleration

- differentiate to go from displacement to velocity to acceleration;
- use **integration** 积分 to go back, from acceleration to velocity to displacement, and use the starting conditions to find each constant of **integration**.
- Integrate: $v = k\left(3t - \frac{t^3}{3}\right)$. The constant is 0, because $v = 0$ when $t = 0$.

Kinematics with variable acceleration (2)

- $k(4.5 - 1.125) = 13.5$, so $3.375k = 13.5$ and $k = 4$.
- Integrate again: $s = 6t^2 - \frac{t^4}{3}$, again with constant 0.
- $v = \frac{4t}{3}(9 - t^2) = 0$ when $t = 3$, and then $s = 54 - 27 = 27$ m.

Kinematics with variable acceleration (3)

- The particle returns to O when $s = 0$, at $t = \sqrt{18}$. It travels 27 m out and 27 m back, so the total distance is 54 m.
- Differentiate: $\mathbf{v} = 2t^{-1/2}\,\mathbf{i} - 3\,\mathbf{j}$, so when $t = 4$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.
- Impulse = change in momentum: $3(\mathbf{v}' - (\mathbf{i} - 3\mathbf{j})) = 6\mathbf{i} - 3\mathbf{j}$, so $\mathbf{v}' = 3\mathbf{i} - 4\mathbf{j}$.

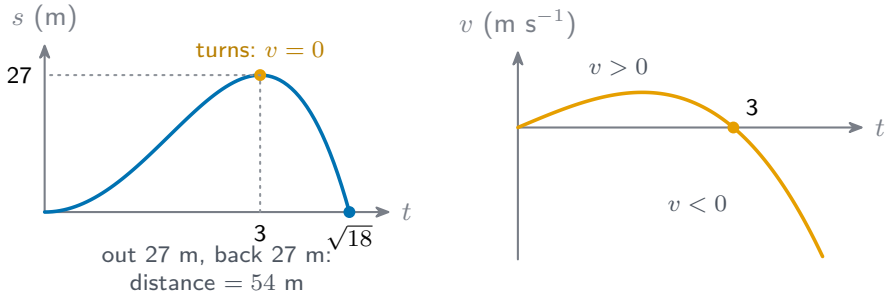
Kinematics with variable acceleration (4)

- The speed is $\sqrt{9+16} = 5$ m s^{-1}.
- horizontally, there is no acceleration, so the velocity stays $u\cos\alpha$;
- vertically, the acceleration is g downwards, so use suvat with initial velocity $u\sin\alpha$.

Kinematics with variable acceleration (5)

- $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, so the initial components are 16 m s^{-1} horizontally and 12 m s^{-1} vertically.
- Vertically, $s = 0$ when it lands: $0 = 12t - 4.9t^2$, so $t = \frac{12}{4.9} = 2.4$ s (2 s.f.).
- Range: $16 \times 2.449 = 39$ m (2 s.f.).
- Greatest height: $0 = 12^2 - 2(9.8)h$, so $h = \frac{144}{19.6} = 7.3$ m (2 s.f.).

Figure: Kinematics with variable acceleration



- The particle stops at t = 3, 27 m from O, then comes back: the total distance is $27 + 27 = 54$ m, while the displacement at return is 0
- Match each part of the diagram to the text.

Centres of mass: particles, laminas and wire frameworks

- For particles of masses $m_1, m_2, \ldots$ at $x_1, x_2, \ldots$, the **centre of mass 质心** is at
- and in the same way for $\overline{y}$.
- For a uniform composite body, use a table:
- 1. split the body into pieces whose centres of mass you know (the booklet gives arcs, sectors, semicircles and triangles); 2. use area as the "mass" of a lamina, and length as the "mass" of a wire; 3. take moments about a convenient line.

$$\overline{x} = \frac{\sum mx}{\sum m}$$

Suspending laminas and frameworks

- G lies on OE (the line of symmetry), $\frac{6a}{3\pi+2}$ from O, and $OA = 2a$. So $\tan \angle OAG = \dfrac{3}{3\pi + 2}$.
- $OA = OE$, so triangle AOE is isosceles and right-angled, and $\angle OAE = 45°$.
- AG is vertical, so the angle between AE and the vertical is $\theta = 45° - \angle OAG$.

Suspending laminas and frameworks (2)

- $\tan\theta = \dfrac{1 - \frac{3}{3\pi+2}}{1 + \frac{3}{3\pi+2}} = \dfrac{3\pi - 1}{3\pi + 5}$, so $\theta = 30.3°$.
- With O as origin and OA along the x-axis: the rectangle has area 24 and centre $(3, 2)$; the square has area 4 and centre $(5, 3)$.
- $\bar{x} = \dfrac{24 \times 3 - 4 \times 5}{20} = 2.6$ cm and $\bar{y} = \dfrac{24 \times 2 - 4 \times 3}{20} = 1.8$ cm.
- OG is vertical, so the angle between OA and the vertical is $\tan^{-1}\frac{1.8}{2.6} = 34.7°$.

Work, energy and power

- The **work-energy principle 功能原理**: the **work done** by all the forces other than gravity equals the change in **kinetic energy** plus **potential energy**.
- **Conservation of mechanical energy 机械能守恒**: if no forces other than gravity do work, KE + PE stays the same.
- **Power 功率** is the rate of doing work. For a vehicle with driving force F moving at speed v, $P = Fv$. At a steady speed, the driving force equals the total of the forces against the motion.

Work, energy and power (2)

- At the top speed V, the forces balance: $\frac{20\,000}{V} = 750 \times 9.8 \times \frac{1}{7} + 200 = 1250$, so $V = 16$ m s^{-1}.
- At 10 m s^{-1}, the driving force is 2000 N: $2000 - 1250 = 750a$, so $a = 1$ m s^{-2}.

Work, energy and power (3)

- The engine is then switched off, at 10 m s^{-1}. The car moves a distance d up the road before it stops: $\frac{1}{2} \times 750 \times 10^2 = 750g \times d\sin\alpha + 200d$, so $37\,500 = 1250d$ and $d = 30$ m. The resistance acts over the distance d along the road, and gravity over the height $d\sin\alpha$.
- Energy per unit mass: $\frac{1}{2}(20^2) = \frac{1}{2}(V^2 + 4V^2) + 10 \times 11$, so $200 = 2.5V^2 + 110$ and $V = 6$. The horizontal component, 12, never changes.

Work, energy and power (4)

- The velocity is at right angles to a $60°$ plane, so it points $30°$ below the horizontal. Its speed is $\frac{12}{\cos 30°} = 8\sqrt{3}$.
- Energy: $200 = \frac{1}{2}(8\sqrt{3})^2 + 10H = 96 + 10H$, so $H = 10.4$ m.

Collisions and restitution

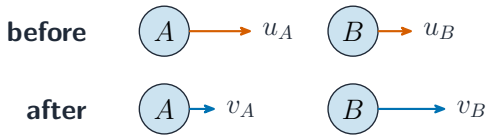
- $e = 1$: perfectly elastic, no **kinetic energy** is lost;
- $e = 0$: the spheres stay together.
- Momentum: $2mu = 2mv_P + mv_Q$. Restitution: $v_Q - v_P = eu$.

$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{v_B - v_A}{u_A - u_B}$$

Collisions and restitution (2)

- Solving: $v_P = \dfrac{(2-e)u}{3}$ and $v_Q = \dfrac{2(1+e)u}{3}$.
- The **kinetic energy** after is $\frac{1}{2}(2m)v_P^2 + \frac{1}{2}mv_Q^2 = \dfrac{mu^2}{3}(2+e^2)$. Before, it was mu^2, so the loss is $\dfrac{mu^2}{3}(1-e^2)$.
- If the impulse on Q is $\frac{8}{9}mu$, then $m \times \dfrac{2(1+e)u}{3} = \dfrac{8mu}{9}$, so $e = \frac{1}{3}$, $v_Q = \frac{8}{9}u$ and $v_P = \frac{5}{9}u$.

Figure: Collisions and restitution



$e = \dfrac{\text{separation speed}}{\text{approach speed}} = \dfrac{v_B - v_A}{u_A - u_B}$

$e = 1$ elastic; $e = 0$ they coalesce;

momentum + restitution: two equations

- Newton's experimental law: separation speed = e $\times$ approach speed
- Match each part of the diagram to the text.

Successive impacts

- $f \times \frac{8}{9}u = \frac{5}{9}u$, so $f = \frac{5}{8}$.
- P and Q now meet head-on with equal speeds. Take the direction towards the wall as positive: P has velocity $\frac{5}{9}u$ and Q has $-\frac{5}{9}u$.
- Momentum: $2m\left(\frac{5}{9}u\right) - m\left(\frac{5}{9}u\right) = 2mv_P + mv_Q$, so $2v_P + v_Q = \frac{5}{9}u$.

Successive impacts (2)

- Restitution with $e = \frac{1}{3}$: $v_Q - v_P = \frac{1}{3} \times \frac{10}{9} u = \frac{10}{27} u$.
- Solving: $v_P = \frac{5}{81} u$ and $v_Q = \frac{35}{81} u$. Both move towards the wall, so Q hits the wall again and rebounds at $\frac{5}{8} \times \frac{35}{81} u = 0.27u$ towards P, which is still moving towards the wall. There is a further collision.

Statics of rigid bodies

- Moments about A: $S \times 1.5a = mg \times a\cos\theta$, so $S = \frac{2}{3}mg\cos\theta$.
- Vertically: $R = mg - S\cos\theta = \frac{mg}{3}(3 - 2\cos^2\theta)$.
- Horizontally: $F = S\sin\theta$, and $F = \frac{4}{7}R$, so $\frac{4}{7} \times \frac{mg}{3}(3 - 2\cos^2\theta) = \frac{2}{3}mg\sin\theta\cos\theta$.

Statics of rigid bodies (2)

- Divide by $\cos^2\theta$ and use $\sec^2\theta = 1 + \tan^2\theta$: $4(1 + 3\tan^2\theta) = 14\tan\theta$, so $6\tan^2\theta - 7\tan\theta + 2 = 0$.
- This factorises as $(2\tan\theta - 1)(3\tan\theta - 2) = 0$, so $\tan\theta = \frac{1}{2}$ or $\frac{2}{3}$.

Sources for this lesson

- Pearson Edexcel, *International Advanced Level Mathematics, Further Mathematics and Pure Mathematics specification* (2018), unit M2.
- I. Newton, *Philosophiae Naturalis Principia Mathematica* (London, 1687), Book 1, Scholium to the Laws of Motion.
- Pearson Edexcel, WME02/01 June 2025, questions 1 to 7 (adapted).

Common errors

- Using suvat when the acceleration is not constant.
- Forgetting the constant of **integration**, or not using the starting conditions to find it.
- Giving the displacement when the question asks for the total distance.

Common errors (2)

- In projectiles, using g in the horizontal motion.
- Using area for a wire, or length for a lamina, in a **centre of mass** table.
- Forgetting the negative sign for a hole or for a piece on the other side of the line.

Common errors (3)

- In energy questions, using the height for the resistance instead of the distance along the slope.
- Writing the restitution equation with the speeds the wrong way round; it is always separation over approach.

Key terms

- **Projectile**: a particle moving freely under gravity.
- **Centre of mass**: the point where the whole weight of a body can be taken to act.
- **Work-energy principle**: the **work done** by non-gravity forces equals the change in mechanical energy.

Key terms (2)

- **Power**: the rate of doing work; $P = Fv$ for a vehicle.
- **Coefficient of restitution**: the ratio of the speed of separation to the speed of approach.
- **Limiting equilibrium**: the state of a body that is on the point of slipping, so $F = \mu R$.

Check yourself

- 1.
- A particle has velocity $v = 6t^2 - 4t$ m s^{-1} and starts at O.
- Find its displacement when $t = 2$. 2.
- A stone is thrown horizontally at 15 m s^{-1} from the top of a cliff 40 m high.