Pure Mathematics 3

Edexcel IAL P3 (WMA13/01); functions, modulus, trigonometry, exponentials, calculus rules, numerical methods

Pure Mathematics 3: functions, identities and calculus at speed

- simplify rational expressions, and work with functions, inverses and the **modulus**;
- use sec, cosec and cot, compound and **double angle** formulae and the **harmonic form**;
- solve equations with e^x and $\ln x$, and fit models using logarithmic graphs;

Pure Mathematics 3: functions, identities and calculus at speed (2)

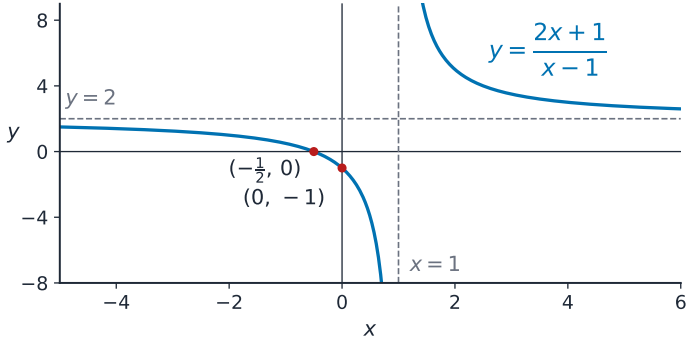
- differentiate with the product, quotient and chain rules;
- integrate standard functions and spot integrals by recognition;
- find roots by **sign change** and by **iteration**.

Algebra: rational expressions and functions

- At $x = 0$: $f(0) = 0$; as $x \to \infty$: $f(x) \to \frac{2}{3}$ from below, and f is increasing (larger x gives larger $f(x)$).
- Let $y = \dfrac{2x}{3x+1}$ and solve for x:
- Check: $f\left(\frac{x}{2-3x}\right) = \dfrac{2x/(2-3x)}{3x/(2-3x)+1} = \dfrac{2x}{3x+2-3x} = x$ ✓.
- Check: $f(\frac{1}{3}) = \frac{2/3}{2} = \frac{1}{3} = f^{-1}(\frac{1}{3})$ ✓.

$$0 \le f(x) < \tfrac{2}{3}$$

Figure: Algebra: rational expressions and functions



- Asymptotes first, then intercepts, then the two branches
- Read the axes and units first.

The modulus function and combined transformations

- Case $2x-1 \geq 0$ ($x \geq \frac{1}{2}$): $2x-1 = x+5 \implies x = 6$ ✓ (satisfies $x \geq \frac{1}{2}$).
- Case $2x-1 < 0$: $-(2x-1) = x+5 \implies -3x = 4 \implies x = -\frac{4}{3}$ ✓ ($< \frac{1}{2}$).
- Both solutions are $x = 6$ and $x = -\frac{4}{3}$. Check: $|2(6)-1| = 11 = 6+5$ ✓; $|-\frac{8}{3}-1| = \frac{11}{3} = -\frac{4}{3}+5$ ✓.

Further trigonometry: sec, cosec, cot and harmonic form

- $R = \sqrt{7^2 + 24^2} = \sqrt{625} = 25.$
- Expand $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$ and match: $25\cos\alpha = 7$ and $25\sin\alpha = 24$, so $\tan\alpha = \frac{24}{7}$ and
- Check: $25\cos(0 + 1.287) = 25(0.28) = 7.0$ ✓ and $-25\sin(1.287) = -24.0$ ✓.

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Further trigonometry: sec, cosec, cot and harmonic form (2)

- Write $3x = 2x + x$ and use the **compound angle 复合角** formula:
- Substitute:
- Check at $x = \frac{\pi}{6}$ ($t = \frac{1}{\sqrt{3}}$): numerator $\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}$; denominator $1 - 1 = 0$ — and $\tan\frac{\pi}{2}$ is undefined ✓ consistent.

$$\alpha = 1.287 \text{ (3 d.p.)} \qquad f(x) = 25\cos(x + 1.287)$$

Exponentials, logarithms and log-graph models

- Gradient: $\dfrac{0-(-2)}{-3.5-0} = -\dfrac{4}{7}$; intercept -2:
- Convert: $y = 10^{-\frac{4}{7}\log_{10} x - 2} = 10^{-2}\, x^{-4/7}$, so
- Check with a point on the line: $\log_{10} x = 0 \Rightarrow x = 1$, $\log_{10} y = -2 \Rightarrow y = \frac{1}{100}$ ✓ and the formula gives $y = \frac{1}{100} \cdot 1 = \frac{1}{100}$ ✓.

$$\log_{10} y = -\tfrac{4}{7}\log_{10} x - 2$$

Exponentials, logarithms and log-graph models (2)

- At $t = 0$: $75 = 21 + A$, so $A = 54$.
- At $t = 5$: $25 = 21 + 54e^{-5k} \implies e^{-5k} = \frac{4}{54} = \frac{2}{27}$:
- Rate: $\dfrac{d\theta}{dt} = -54ke^{-kt}$; set equal to -9:

$$y = \frac{1}{100}\,x^{-4/7} \qquad (p = \tfrac{1}{100},\ q = -\tfrac{4}{7})$$

Exponentials, logarithms and log-graph models (3)

- The **initial value** 初值 is 75 °C and the **long-run floor** 长期下限 is 21 °C.
- Check: at $T = 2.19$, $\theta = 21 + 54e^{-0.521(2.19)} = 21 + 54(0.320) = 38.3$ °C, falling at $28.13 \times 0.320 = 9.0$ °C/min ✓. The **model**'s floor is 21 °C (room temperature) — sensible for large t.

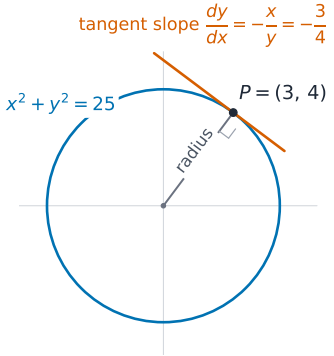
$$-5k = \ln \tfrac{2}{27} \implies k = \tfrac{1}{5} \ln \tfrac{27}{2} = 0.521 \text{ (3 s.f.)}$$

Differentiation: rules, implicit and models

- **Product rule** with $u = e^{-x^2}$ (chain: $u' = -2xe^{-x^2}$) and $v = \sin 3x$ ($v' = 3\cos 3x$):
- Check at $x = 0$: derivative $= 1 \times (3 - 0) = 3$, and near 0 the curve behaves like $\sin 3x$ with gradient 3 ✓.

product: $(uv)' = u'v + uv'$ quotient: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ chain: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Figure: Differentiation: rules, implicit and models



- **Implicit differentiation** gives the tangent to a circle, perpendicular to the radius
- Match each part of the diagram to the text.

Integration: standard forms and recognition

- **Double angle 二倍角** with $A = 3x$: $\cos 6x \equiv 1 - 2\sin^2 3x$, so $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$:
- Check by differentiating: $\frac{1}{2} - \frac{6\cos 6x}{12} = \frac{1}{2}(1 - \cos 6x) = \sin^2 3x$ ✓.
- The numerator is the derivative of the denominator ($f' = 6x$, $f = 3x^2 + 4$):
- Check: $\dfrac{d}{dx}\ln(3x^2 + 4) = \dfrac{6x}{3x^2 + 4}$ ✓. The powered form works the same way: with $u = 3x^2 + 4$ and $du = 6x\,dx$,

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c \qquad \int f'(x)\big(f(x)\big)^n\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1} + c \quad (n \neq -1)$$

Figure: Integration: standard forms and recognition

$$\int \frac{\overbrace{6x}^{f'(x)}}{\underbrace{3x^2+4}_{f(x)}}\,dx = \ln|3x^2+4| + c$$

numerator a multiple of the inner derivative? then check by differentiating

$$\int x\,(3x^2+4)^4\,dx = \frac{(3x^2+4)^5}{30} + c$$

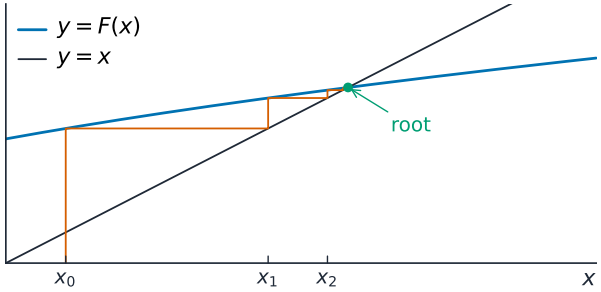
- Spot f prime beside f
- Match each part of the diagram to the text.

Numerical methods: sign changes and iteration

- **Sign change 变号**: if f is continuous on $[a, b]$ and $f(a)f(b) < 0$, a root lies between a and b. Halve the interval to pin a root to any accuracy.
- **Iteration 迭代**: a rearrangement $x_{n+1} = f(x_n)$ from a start value converges to a **fixed point 不动点** (or not — always check against a sign-change bound).
- Check the **iteration** converges: $|x_3 - x_2|$ small and the values settle toward 0.43 ✓.
- g is continuous, and the **sign change** between 0.4295 and 0.4305 brackets a root, so the root is 0.430 to 3 d.p. ✓.

$$x_2 = \tfrac{1}{3}\arctan\left(\tfrac{3}{0.8}\right) = \tfrac{1}{3}\arctan(3.75) = 0.4367 \qquad x_4 = 0.4307 \text{ (4 d.p.)}$$

Figure: Numerical methods: sign changes and iteration



$x_{n+1} = F(x_n)$ · converges near the root if $|F'(x)| < 1$ there

- Each step goes up to $y = F(x)$ then across to $y = x$; the staircase closes in on the root.
- Read the axes and units first.

Sources for this lesson

- Pearson Edexcel, *International Advanced Level Mathematics, Further Mathematics and Pure Mathematics specification* (Issue 3, 2019), unit P3.
- E. Maor, *e: The Story of a Number* (Princeton University Press, 1994), chapters 3 and 4.
- Pearson Edexcel, WMA13/01 June 2025, questions 2 to 8 (adapted).

Common errors

- Cancelling terms instead of factors in a **rational expression**.
- Writing $f^{-1}(x)$ as $\frac{1}{f(x)}$; the **inverse function** undoes f, it is not a reciprocal.
- Forgetting to check solutions of a **modulus** equation in the original equation.

Common errors (2)

- In **harmonic form**, finding $\tan\alpha$ the wrong way up.
- Writing $\ln(a+b) = \ln a + \ln b$.
- Forgetting the **chain rule** factor, for example $\frac{d}{dx}e^{3x} = 3e^{3x}$, not e^{3x}.

Common errors (3)

- Integrating $\frac{1}{x}$ as $\frac{x^0}{0}$ instead of $\ln|x|$.
- Stating a root from a **sign change** without saying the **function** is continuous on the interval.

Key terms

- **Inverse function**: the **function** f^{-1} that reverses f; its graph is the reflection of $y = f(x)$ in $y = x$.
- **Modulus**: the size of a number without its sign, $|x|$.
- **Harmonic form**: $a\cos\theta + b\sin\theta$ written as $R\cos(\theta - \alpha)$.

Key terms (2)

- **Chain rule**: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
- **Iteration**: repeatedly using a formula $x_{n+1} = g(x_n)$ to get closer to a root.
- **Sign change**: a change in the sign of $f(x)$ between two values, which shows a root between them if f is continuous.

Check yourself

- 1.
- $f(x) = 3x - 2$.
- Find $f^{-1}(x)$. 2.
- Solve $|x - 3| = 5$. 3.