Pure Mathematics 2

Edexcel IAL P2 (WMA12/01); proof, factor theorem, circles, series, logs, trigonometry, calculus

Pure Mathematics 2: building on the P1 toolkit

- prove results by exhaustion and disprove them with a counterexample;
- use the factor and remainder theorems and **algebraic division**;
- find equations of circles and use their geometry;

Pure Mathematics 2: building on the P1 toolkit (2)

- work with arithmetic and geometric series and the **binomial expansion**;
- solve exponential, logarithmic and **trigonometric equations**;
- find **stationary points**, areas and trapezium rule estimates.

Proof

- **Proof by exhaustion** 穷举法: split the possibilities into cases and check every case.
- **Disproof by counterexample** 反例法: one example that breaks the claim disproves it.
- Algebraic **proof**: start from what is given, manipulate identities (not equations to solve), and end with the target statement.

Proof (2)

- Factorise first: $m^2 + 3m + 2 = (m + 1)(m + 2)$.
- A positive integer not divisible by 3 is either $m = 3q + 1$ or $m = 3q + 2$ — those are all the cases, so exhausting them proves the claim.
- Case $m = 3q + 1$: $(m + 1)(m + 2) = (3q + 2)(3q + 3) = 3(3q + 2)(q + 1)$ — a multiple of 3.

Proof (3)

- Case $m = 3q + 2$: $(m+1)(m+2) = (3q+3)(3q+4) = 3(q+1)(3q+4)$ — a multiple of 3.
- Both cases give a multiple of 3, so $3 \mid m^2 + 3m + 2$ whenever $3 \nmid m$. ■
- Check: $m = 1$: $6 = 3 \times 2$ ✓; $m = 2$: $12 = 3 \times 4$ ✓; $m = 4$: 30 ✓; $m = 5$: 42 ✓.

Proof (4)

- One counterexample is enough. Try $n = 5$: $25 - 5 + 1 = 21 = 3 \times 7$, which is not prime.
- The claim is false. Check $n = 2, 3, 4$ give the primes $3, 7, 13$ — the pattern holds at first, which is exactly why a counterexample, not more examples, settles the question.

Algebra: the factor and remainder theorems

- **Factor theorem 因式定理**: $(x-b)$ is a factor of $f(x)$ exactly when $f(b)=0$. For divisors of the form $(ax-b)$: if $f\left(\frac{b}{a}\right)=0$ then $(ax-b)$ is a factor.
- **Remainder theorem 余项定理**: dividing $f(x)$ by $(ax-b)$ leaves remainder $f\left(\frac{b}{a}\right)$.
- **Algebraic division 代数除法** by $(ax\pm b)$ gives a **quotient 商** and a **remainder**; the degree drops by one: $f(x)=(ax-b)\times(\text{quotient})+\text{remainder}$.

$$a(-27)+b(9)+5(-3)-3=0 \implies -27a+9b-18=0 \implies b=3a+2$$

Algebra: the factor and remainder theorems (2)

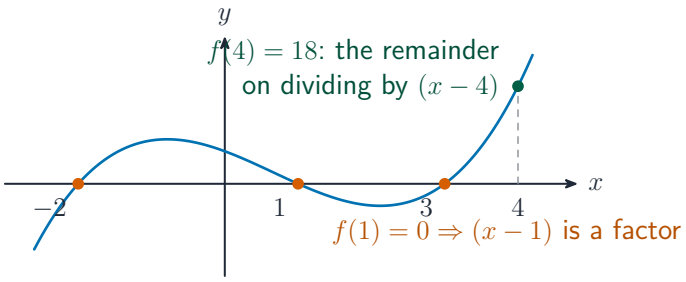
- **Factor theorem**: $f(-3) = 0$:
- **Remainder theorem** with $(2x - 1)$: $f\left(\frac{1}{2}\right) = 7$:
- Substitute $b = 3a + 2$: $a + 6a + 4 = 60$, so $a = 8$ and $b = 26$.

$$\frac{a}{8} + \frac{b}{4} + \frac{5}{2} - 3 = 7 \implies \frac{a}{8} + \frac{b}{4} = 7.5 \implies a + 2b = 60$$

Algebra: the factor and remainder theorems (3)

- Check: $f(x) = 8x^3 + 26x^2 + 5x - 3$; $f(-3) = -216 + 234 - 15 - 3 = 0$ ✓ and $f(\frac{1}{2}) = 1 + 6.5 + 2.5 - 3 = 7$ ✓.
- Divide $8x^3 + 26x^2 + 5x - 3$ by $(x - 2)$: **quotient** $8x^2 + 42x + 89$, remainder 175.
- Check by rebuilding: $(x - 2)(8x^2 + 42x + 89) + 175 = 8x^3 + 42x^2 + 89x - 16x^2 - 84x - 178 + 175 = 8x^3 + 26x^2 + 5x - 3$ ✓. The remainder also equals $f(2) = 64 + 104 + 10 - 3 = 175$ ✓.

Figure: Algebra: the factor and remainder theorems



- Roots are factors; the height at x = b is the remainder on dividing by (x − b)
- Read the axes and units first.

Coordinate geometry of the circle

- the angle in a semicircle is a right angle;
- the perpendicular from the centre to a **chord** 弦 bisects the **chord**;
- a **tangent** 切线 is perpendicular to the radius at the point of contact.

$$(x-a)^2+(y-b)^2=r^2$$

Coordinate geometry of the circle (2)

- The centre is the midpoint: $\left(\frac{-2+4}{2}, \frac{5+15}{2}\right) = (1, 10)$.
- The diameter squared: $d^2 = (4-(-2))^2 + (15-5)^2 = 36 + 100 = 136$, so $r^2 = 136 \div 4 = 34$.
- Check: both endpoints satisfy the equation: $(-3)^2 + (-5)^2 = 34$ ✓ and $3^2 + 5^2 = 34$ ✓.

$$(x-1)^2 + (y-10)^2 = 34$$

Coordinate geometry of the circle (3)

- The nearest point lies on the vertical radius: same x as the centre, r below it:
- Check: $(1-1)^2 + (10-\sqrt{34}-10)^2 = 34$ ✓, and no smaller y is possible since the centre sits $10 > \sqrt{34}$ above the axis.

$$(1,\ 10-\sqrt{34})$$

Sequences and series

- $a = 10$, $r = 8/10 = 0.8$, and $|r| < 1$:
- The kth term is $10(0.8)^{k-1}$:
- Take logs (the log of a number below 1 is negative — keep the inequality direction when dividing by it):

$$a_n = a + (n-1)d \qquad S_n = \frac{n}{2}\big(2a + (n-1)d\big) \qquad \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Sequences and series (2)

- So $k > 45.38$ and the smallest integer is $k = 46$.
- Check: $10(0.8)^{45} = 0.000436\ldots < 0.0005$ ✓ but $10(0.8)^{44} = 0.000544\ldots > 0.0005$ ✓.
- Terms stay positive until $u_n = 850 - 7(n-1) > 0$, so $n \leq 122.4$; the 122nd term is $850 - 7(121) = 3 > 0$ and the 123rd is $-4 < 0$.
- Check: adding a negative 123rd term would reduce the sum, so S_{122} is the **maximum** ✓.

$$a_n = ar^{n-1} \qquad S_n = \frac{a(1-r^n)}{1-r} \qquad S_\infty = \frac{a}{1-r}\ (|r| < 1)$$

Exponentials and logarithms

- Bring everything to one log per side:
- Equal logs give equal arguments:
- **Domain** check: $4 - x > 0$ needs $x < 4$, so $x = 14$ is rejected. $x = -2$**.**

$$\log_a x + \log_a y \equiv \log_a(xy) \qquad \log_a x - \log_a y \equiv \log_a \frac{x}{y} \qquad k\log_a x \equiv \log_a(x^k) \qquad \log_a a = 1$$

Exponentials and logarithms (2)

- Check: LHS $= 2\log_2 6$; RHS $= 3 + \log_2(4.5) = \log_2(8 \times 4.5) = \log_2 36 = 2\log_2 6$ ✓.
- Set equal and substitute $u = 3^x$:
- $P = (\log_3 2,\ 12)$, so $a = 2$, $b = 12$. Check: $3^{2\log_3 2+1} = 3 \cdot 3^{2\log_3 2} = 3 \cdot 4 = 12$ ✓.

$$\log_2(4-x)^2 = \log_2 8 + \log_2\left(\frac{x+11}{2}\right) = \log_2\left(4(x+11)\right)$$

Trigonometric identities and equations

- Divide by 3: $\tan\left(2x+\frac{\pi}{5}\right)=\frac{1}{\sqrt{3}}$, and $\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$:
- In $0\le x<\pi$: $x=\frac{29\pi}{60}$ (k $=1$) and $x=\frac{59\pi}{60}$ (k $=2$).
- Check: at $x=\frac{29\pi}{60}$: $2x+\frac{\pi}{5}=\frac{7\pi}{6}$ and $\tan\frac{7\pi}{6}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$ ✓.

$$\tan\theta\equiv\frac{\sin\theta}{\cos\theta}\qquad\sin^2\theta+\cos^2\theta\equiv1$$

Trigonometric identities and equations (2)

- Replace $\tan\theta = \sin\theta/\cos\theta$ and multiply through by $\cos\theta$:
- Use $\sin^2\theta = 1 - \cos^2\theta$:
- Quadratic formula in $\cos\theta$:

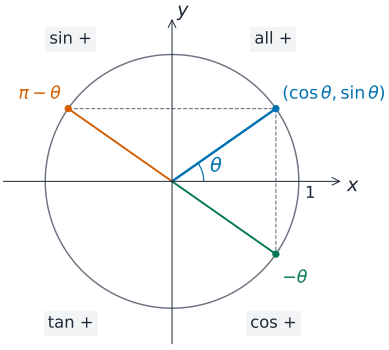
$$2x + \frac{\pi}{5} = \frac{\pi}{6} + k\pi \implies 2x = -\frac{\pi}{30} + k\pi \implies x = -\frac{\pi}{60} + \frac{k\pi}{2}$$

Trigonometric identities and equations (3)

- The $-$ root is below -1 (reject); $\cos\theta = 0.6385$ gives $\theta = 50.3°$ and, from cosine's symmetry, $\theta = 360° - 50.3° = 309.7°$.
- Check: $\cos 50.3° = 0.638$ ✓; both angles lie in the interval ✓.

$$\frac{5\sin^2\theta}{\cos\theta} = \cos\theta + 4 \implies 5\sin^2\theta = \cos^2\theta + 4\cos\theta$$

Figure: Trigonometric identities and equations



- Reflections in the unit circle give the symmetry rules
- Match each part of the diagram to the text.

Differentiation: stationary points

- derivative changes $+$ to $-$: **maximum** 极大值;
- derivative changes $-$ to $+$: **minimum** 极小值;
- the second derivative helps: $f''(x) > 0$ at the point confirms a **minimum**, $f''(x) < 0$ a **maximum**.

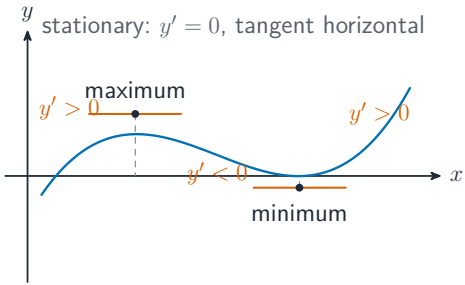
$$S = 2 \times \tfrac{1}{2}(3x)(2x) + 3xL + 2xL = 6x^2 + 5x \cdot \frac{4}{x^2} = 6x^2 + \frac{20}{x}$$

Differentiation: stationary points (2)

- Volume: cross-section $\times$ length $= \frac{1}{2}(3x)(2x) \times L = 3x^2L = 12$, so $L = \dfrac{4}{x^2}$.
- Surface: two triangles plus the two rectangles:
- Positive second derivative everywhere on the **domain**: the stationary point is a **minimum** ✓.
- Check: at $x = 1.1856$, $\frac{d^2S}{dx^2} = 12 + 40/(5/3) = 12 + 24 = 36 > 0$ ✓.

$$\frac{dS}{dx} = 12x - \frac{20}{x^2} = 0 \implies 12x^3 = 20 \implies x^3 = \frac{5}{3} \implies x = 1.1856\ldots$$

Figure: Differentiation: stationary points



- Stationary means the **tangent** is horizontal; the sign change classifies it.
- Read the axes and units first.

Integration: areas and the trapezium rule

- Line PQ: gradient $\dfrac{5-11}{4-1} = -2$, so $y - 11 = -2(x-1) \implies y = -2x + 13$.
- Check Q: $-8 + 13 = 5$ ✓.
- Area under the line: $\displaystyle\int_1^4 (-2x+13)\,dx = \left[-x^2 + 13x\right]_1^4 = 36 - 12 = 24.$

$$\int_a^b y\,dx = F(b) - F(a) \quad \text{where } F'(x) = y$$

Integration: areas and the trapezium rule (2)

- **Area under the curve**:
  $\int_1^4 \left(3x + 16x^{-2} - 8\right) dx = \left[\frac{3}{2}x^2 - \frac{16}{x} - 8x\right]_1^4 = (24 - 4 - 32) - (\frac{3}{2} - 16 - 8) = 10.5.$
- Region area: $24 - 10.5 = \mathbf{13.5}$ square units.
- Check: the line is above the curve throughout (at $x = 2$: line 9, curve $3(2) + 4 - 8 = 2$), so subtracting curve from line is the right way round ✓.

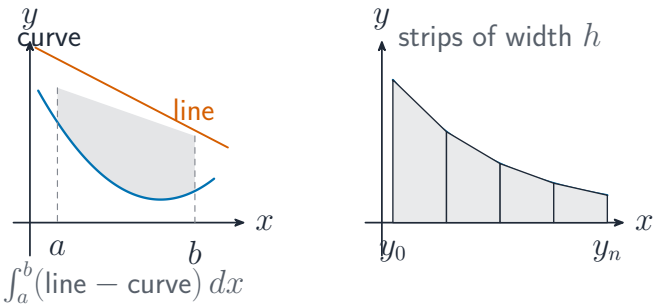
$$\int_a^b y\,dx \approx h\left[\tfrac{1}{2}(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1}\right] \quad \text{or} \quad \frac{h}{2}\left[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})\right]$$

Integration: areas and the trapezium rule (3)

- Four strips over $[1, 3]$ give $h = 0.5$ and five values of $y = \dfrac{1}{x}$: at $x = 1,\ 1.5,\ 2,\ 2.5,\ 3$ the values are $1,\ 0.6667,\ 0.5,\ 0.4,\ 0.3333$.
- The true value is $\ln 3 = 1.099$ (3 s.f.). The estimate is an **overestimate**: $y = 1/x$ is convex (it bends upwards), so every straight strip sits above the curve ✓.

$$\int_1^3 \frac{1}{x}\,dx \approx \frac{0.5}{2}\big[1 + 0.3333 + 2(0.6667 + 0.5 + 0.4)\big] = 0.25 \times 4.4667 = 1.12$$

Figure: Integration: areas and the trapezium rule



- Area between uses $\int(\text{upper} - \text{lower})$, splitting at crossings; the **trapezium rule** joins neighbouring values with straight strips.
- Read the axes and units first.

Sources for this lesson

- Pearson Edexcel, *International Advanced Level Mathematics, Further Mathematics and Pure Mathematics specification* (Issue 3, 2019), unit P2.
- J. Napier, *Mirifici Logarithmorum Canonis Descriptio* (Edinburgh, 1614).
- Pearson Edexcel, WMA12/01 June 2025, questions 2 to 10 (adapted).

Common errors

- Trying to prove a general statement by checking a few examples.
- Using $f(b)$ instead of $f\left(\frac{b}{a}\right)$ for the divisor $(ax-b)$.
- Getting the sign of the centre wrong: $(x-3)^2+(y+2)^2=25$ has centre $(3,-2)$.

Common errors (2)

- Using the sum to infinity when $|r| \geq 1$.
- Forgetting to reverse an inequality when dividing by the **logarithm** of a number less than 1, which is negative.
- Writing $\log(a + b) = \log a + \log b$.

Common errors (3)

- Giving only one solution of a trigonometric equation when the interval contains more.
- Finding a stationary point without showing whether it is a **maximum** or a **minimum**.

Key terms

- **Proof by exhaustion**: proving a statement by checking every possible case.
- **Factor theorem**: $(x - b)$ is a factor of $f(x)$ if and only if $f(b) = 0$.
- **Geometric sequence**: a **sequence** in which each term is the previous term multiplied by a constant ratio r.

Key terms (2)

- **Logarithm**: $\log_a x = y$ means $a^y = x$.
- **Stationary point**: a point on a curve where the gradient is zero.
- **Trapezium rule**: a method of estimating an area under a curve using strips shaped like trapezia.

Check yourself

- 1.
- Show that $(x-2)$ is a factor of $x^3 - 3x^2 + 4$. 2.
- Find the centre and radius of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$. 3.
- An arithmetic series has first term 5 and common difference 3.