Course Companion

Pearson Edexcel · International A-Level · Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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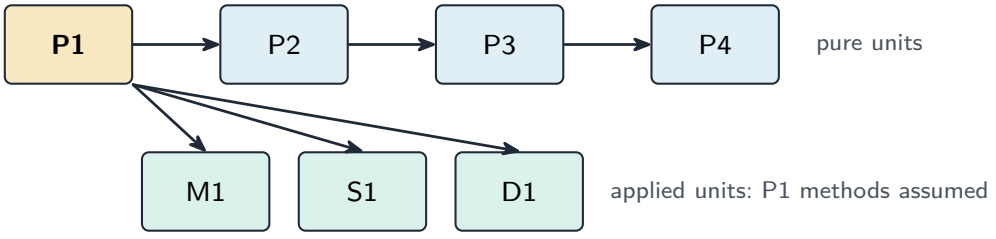
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1 · Pure Mathematics 1

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Pure Mathematics 1: the toolkit every later unit uses

Pure Mathematics 1 (P1) is the first compulsory unit of both International AS and International A Level Mathematics, and of International A Level Pure Mathematics. This reference covers every P1 topic in the Edexcel IAL specification (Issue 3, April 2019): algebra and functions, coordinate geometry, trigonometry, differentiation and integration.



P1 comes first: every later pure and applied unit assumes it

- The P1 paper (WMA11/01) lasts **1 hour 30 minutes** with **75 marks**, and you must answer all questions. Recent papers have ten questions from 5 to 11 marks each.
- A **calculator** 计算器 may be used, but not one with symbolic algebra, symbolic differentiation or integration, or retrievable stored formulae.
- The yellow booklet *Mathematical Formulae and Statistical Tables* is provided. The formulae in the "you must know" table below do **not** appear in that booklet, so memorise them.
- Give inexact answers to **3 significant figures** unless told otherwise. When a question says "you must show all stages of your working", a calculator answer alone scores nothing.

Formulae you must know for P1 (from the specification's assessment information):

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Formula	Use
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	roots of $ax^2 + bx + c = 0$
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	sine rule 正弦定理
$a^2 = b^2 + c^2 - 2bc\cos A$	cosine rule 余弦定理
area $= \frac{1}{2}ab\sin C$	area of a triangle
arc length $s = r\theta$, sector area $= \frac{1}{2}r^2\theta$	θ in radians 弧度
$\dfrac{d}{dx}x^n = nx^{n-1}$	power rule for differentiation
$\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$	power rule for integration

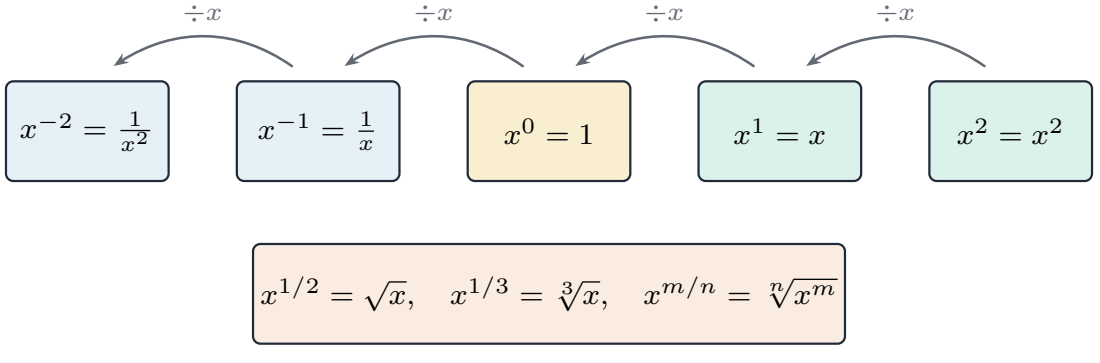
Algebra and functions

Indices and surds

For positive bases, the **laws of indices** 指数运算法则 hold for rational exponents. Zero and negative bases need their real-domain restrictions:

$$a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad (a^m)^n = a^{mn}$$

Fractional powers are roots: $a^{m/n} = \sqrt[n]{a^m}$, so $a^{1/2} = \sqrt{a}$ and $a^{1/3} = \sqrt[3]{a}$. A negative power means reciprocal: $a^{-1} = 1/a$. Questions usually ask you to write an answer **in the form** kx^n with k and n fully simplified constants.



fractional powers are roots; negative powers are reciprocals

Negative powers are reciprocals; fractional powers are roots

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Worked example – three index manipulations (adapted from Edexcel IAL WMA11 June 2025 Q1). Given $p = \dfrac{x^4}{16}$ and $q = \dfrac{40}{3x}$, write each expression in the form kx^n.

(a) $p^{\frac{1}{2}}$.

- Known: $p = x^4/16$, and a power $\frac{1}{2}$ means square root.

$$p^{\frac{1}{2}} = \sqrt{\frac{x^4}{16}} = \frac{\sqrt{x^4}}{\sqrt{16}} = \frac{x^2}{4}$$

- Check: squaring the answer, $(x^2/4)^2 = x^4/16 = p$.

(b) $(pq)^{-1}$.

- Known: multiply before inverting, using $x^4 \cdot x^{-1} = x^3$.

$$pq = \frac{x^4}{16} \times \frac{40}{3x} = \frac{40x^3}{48} = \frac{5x^3}{6}$$

$$(pq)^{-1} = \left(\frac{5x^3}{6}\right)^{-1} = \frac{6}{5}x^{-3}$$

- Check: $(pq) \times (pq)^{-1} = \frac{5x^3}{6} \times \frac{6}{5x^3} = 1$.

(c) $(pq)^2$.

$$(pq)^2 = \left(\frac{5x^3}{6}\right)^2 = \frac{25x^6}{36}$$

- Check: $5^2 = 25$, $6^2 = 36$, $(x^3)^2 = x^6$, and the constants are already fully simplified.

A **surd** 根式 is an exact root such as $\sqrt{2}$. Keep answers exact unless asked for decimals; examiners penalise rounded decimals where a surd is expected. To **rationalise the denominator** 分母有理化, multiply numerator and denominator by a term that removes the root:

Worked example – rationalising. Write $\dfrac{12}{1+\sqrt{2}}$ with a rational denominator.

- Known: $(1+\sqrt{2})(\sqrt{2}-1) = \sqrt{2} - 1 + 2 - \sqrt{2} = 1$, so multiply by $\dfrac{\sqrt{2}-1}{\sqrt{2}-1}$.

$$\frac{12}{1+\sqrt{2}} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{12(\sqrt{2}-1)}{(1+\sqrt{2})(\sqrt{2}-1)} = 12(\sqrt{2}-1) = 12\sqrt{2} - 12$$

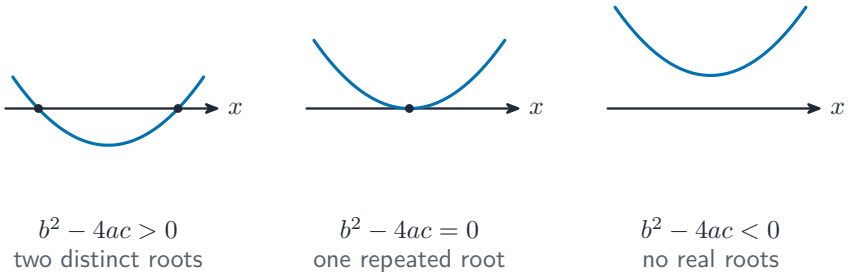
- Check by calculator: $12/(1 + 1.4142\ldots) = 4.9706\ldots$ and $12\sqrt{2} - 12 = 4.9706\ldots$ ✓

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■ Quadratics: discriminant and completing the square

A **quadratic function** 二次函数 has the form $y = ax^2 + bx + c$ with $a \neq 0$; its graph is a parabola opening up ($a > 0$) or down ($a < 0$). The **discriminant** 判别式 $\Delta = b^2 - 4ac$ counts the roots:

- $\Delta > 0$: two distinct real roots – the curve cuts the x-axis twice;
- $\Delta = 0$: one repeated root – the curve touches the x-axis;
- $\Delta < 0$: no real roots – the curve never meets the x-axis.



The discriminant counts intersections with the x-axis.

Completing the square 配方 rewrites ax^2+bx+c as $a\left(x+\dfrac{b}{2a}\right)^2+\left(c-\dfrac{b^2}{4a}\right)$, which gives the turning point $\left(-\dfrac{b}{2a},\, c-\dfrac{b^2}{4a}\right)$ directly. Solve quadratics by factorisation, the formula, completing the square, or a calculator – but "show all stages" questions reject calculator-only working.

> **Worked example – a hidden quadratic** (adapted from Edexcel IAL WMA11 June 2025 Q10). The equation $(k-1)x^6 + 4x^3 + (k-4) = 0$, where k is a constant.

(a) For $k = 4.5$, find the exact solutions.

- Known: with $k = 4.5$ the equation is $3.5x^6 + 4x^3 + 0.5 = 0$; multiply by 2 to clear decimals:
$$7x^6 + 8x^3 + 1 = 0$$
- Why substitute: $x^6 = (x^3)^2$, so this is a quadratic in x^3. Substitute $z = x^3$:
$$7z^2 + 8z + 1 = 0 \implies (7z+1)(z+1) = 0 \implies z = -\tfrac{1}{7} \text{ or } z = -1$$
- Return to x: $x^3 = -\frac{1}{7}$ or $x^3 = -1$, so
$$x = -\sqrt[3]{\tfrac{1}{7}} = -\frac{1}{\sqrt[3]{7}} = -\frac{\sqrt[3]{49}}{7} \qquad \text{or} \qquad x = -1$$
- Check: $z = -1$ gives $7 - 8 + 1 = 0$ ✓; $z = -\frac{1}{7}$ gives $7 \cdot \frac{1}{49} - \frac{8}{7} + 1 = \frac{1}{7} - \frac{8}{7} + 1 = 0$ ✓

(b) Find the set of values of k for which the equation has **no real roots**.

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- Known: treat the equation as a quadratic in x^3 with $a = k-1$, $b = 4$, $c = k-4$. No real roots needs a negative discriminant:

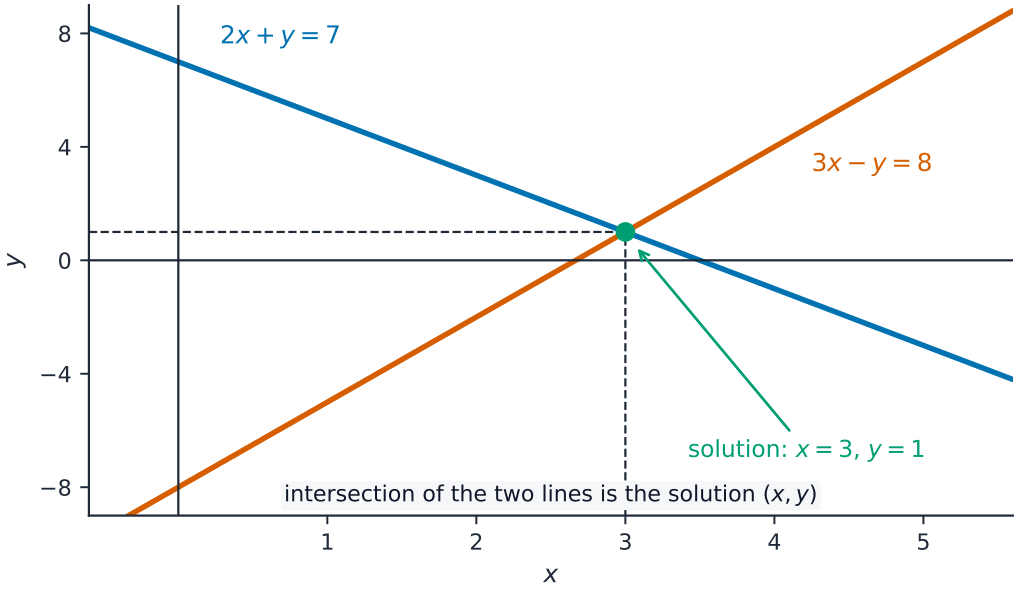
$$b^2 - 4ac = 16 - 4(k-1)(k-4) = 16 - 4(k^2 - 5k + 4) = -4k^2 + 20k = 4k(5-k)$$

$$4k(5-k) < 0 \implies k < 0 \text{ or } k > 5$$

- Check: at $k = -1$: $4(-1)(6) = -24 < 0$ ✓; at $k = 6$: $4(6)(-1) = -24 < 0$ ✓; at $k = 3$: $4(3)(2) = 24 > 0$ ✓ (two real roots there).

■ Simultaneous equations and inequalities

Solve **simultaneous equations** 联立方程 analytically by **substitution** 代入法: rearrange the linear equation, substitute into the other, solve, then substitute back. A line meeting a curve gives a quadratic whose roots are the intersection x-coordinates – this is the "geometrical interpretation of algebraic solutions" the specification asks for.



The solution of simultaneous equations is where their graphs cross

Worked example – line meets cubic-squared curve (adapted from Edexcel IAL WMA11 June 2025 Q7). C_1: $y = 0.2x^2(x-4)$ and C_2: $y = 1.2x(8-x)$ intersect where

$$0.2x^2(x-4) = 1.2x(8-x)$$

- Expand both sides: $0.2x^3 - 0.8x^2 = 9.6x - 1.2x^2$.
- Collect terms: $0.2x^3 + 0.4x^2 - 9.6x = 0$, then divide by 0.2: $x^3 + 2x^2 - 48x = 0$.
- Factorise: $x(x^2 + 2x - 48) = x(x+8)(x-6) = 0$, so $x = 0, -8, 6$.
- Substitute back into C_2: $(0, 0)$, $(-8, -153.6)$, $(6, 14.4)$.

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- Check with C_1: $0.2(-8)^2(-12) = 0.2 \cdot 64 \cdot (-12) = -153.6$ ✓ and $0.2(36)(2) = 14.4$ ✓

Inequalities 不等式 follow the same algebra, but the answer is a range:

- Solve $ax+b > cx+d$ by collecting x (the inequality sign survives adding/subtracting, and survives dividing by a **positive** number; dividing by a negative flips the sign).
- Solve $px^2 + qx + r \leq 0$ by sketching: the solution is where the parabola is on or below the axis: between two roots when $p > 0$, and outside them when $p < 0$. Check repeated/no-real-root cases separately.
- "Curve below line" inequalities such as $px^2 + qx + r < ax + b$ mean: where is the curve's height less than the line's height? Find intersections first.
- When **sketching** the solution region, use a **dotted line** for strict $<$ or $>$ boundaries and a **solid line** for $\leq$ or $\geq$.

> **Worked example.** Solve $x^2 < 3x + 10$.

- Rearrange: $x^2 - 3x - 10 < 0$, and factorise: $(x - 5)(x + 2) < 0$.
- The parabola $y = (x - 5)(x + 2)$ opens upward, so it is below zero **between** its roots:

$$-2 < x < 5$$

- Check: $x = 0$ gives $0 < 10$ ✓; $x = 6$ gives $36 < 28$ × (correctly excluded).

■ Polynomials: expanding and factorising

Expand brackets and collect like terms; factorise by taking out common factors and by grouping. Cubics factorise fully once one root is spotted: $x^3 + 4x^2 + 3x = x(x^2 + 4x + 3) = x(x + 1)(x + 3)$. The notation $f(x)$ names the function, so $f(2)$ means "substitute $x = 2$".

■ Sketching graphs and transformations

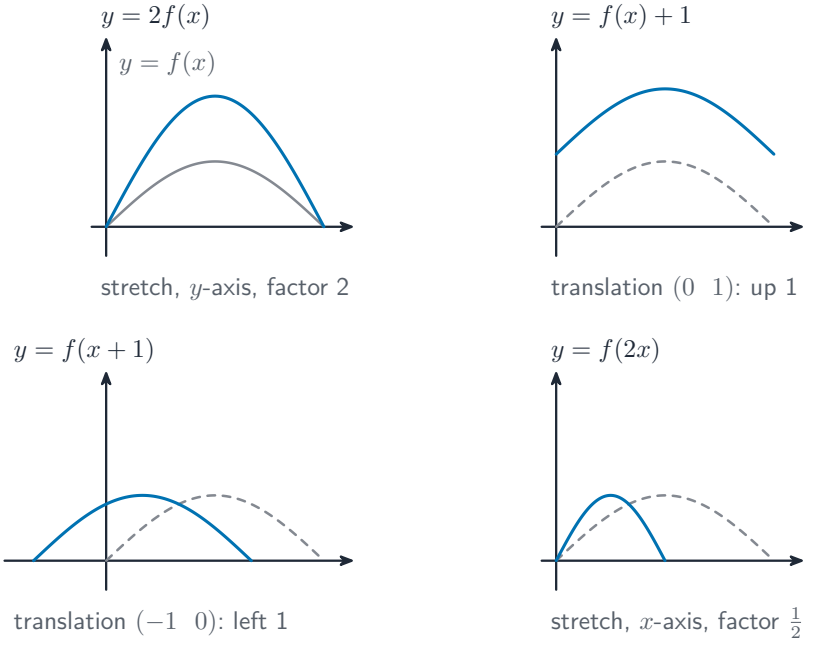
Sketch (do not plot) these families: quadratics and simple **cubics** 三次函数 such as $y = x^3 - 4x$; **reciprocal functions** 反比例函数 $y = \dfrac{k}{x}$ and $y = \dfrac{k}{x^2}$ $(x \neq 0)$, which have an **asymptote** 渐近线 – a line approached in the relevant limit; a horizontal or oblique asymptote may be crossed at a finite input; and the trigonometric graphs below. Mark intercepts and turning points.

Intersections answer equations: the curves $y = f(x)$ and $y = g(x)$ meet where $f(x) = g(x)$.

One **transformation** 变换 at a time, applied to $y = f(x)$:

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New equation	What it does to the graph
$y = af(x)$	stretch parallel to the y-axis, scale factor a
$y = f(x) + a$	translation by $\begin{pmatrix} 0 \\ a \end{pmatrix}$ (moves up by a)
$y = f(x + a)$	translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$ (moves left by a)
$y = f(ax)$	stretch parallel to the x-axis, scale factor $\dfrac{1}{a}$



Name the direction, the type and the scale factor (or vector).

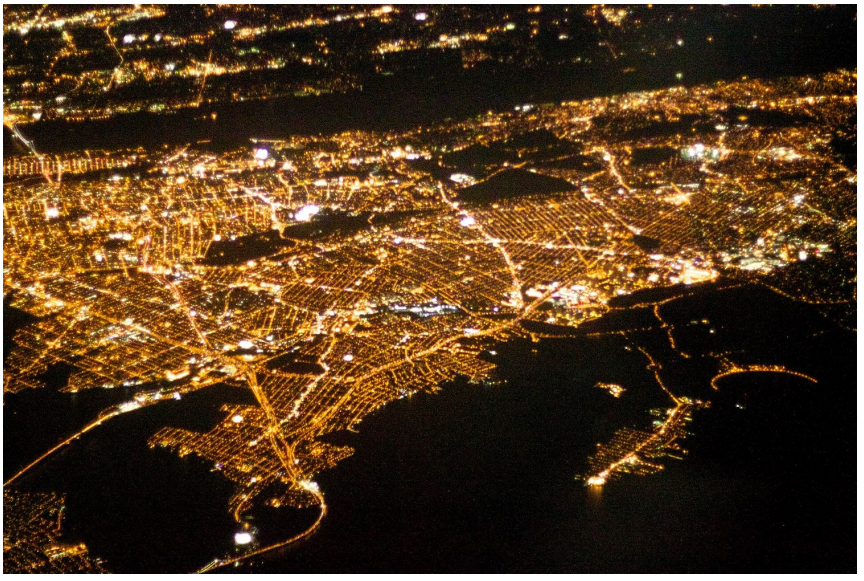
Worked example – describe two transformations (adapted from Edexcel IAL WMA11 June 2025 Q9).

- $y = \sqrt{2x}$ from $y = \sqrt{x}$: write $\sqrt{2x} = f(2x)$, so this is a **stretch parallel to the x-axis with scale factor $\frac{1}{2}$**.
- $y = 12 - \sqrt{x}$ from $y = -\sqrt{x}$: write $12 - \sqrt{x} = -\sqrt{x} + 12$, a **translation by** $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$ (up by 12).
- Check with a point: $y = \sqrt{x}$ passes $(4, 2)$; after the first transformation the curve passes $(2, 2)$ (halved x), and $y = \sqrt{2 \times 2} = 2$ ✓. After the second, $y = -\sqrt{4} + 12 = 10$, so $(4, 10)$ lies on it ✓.

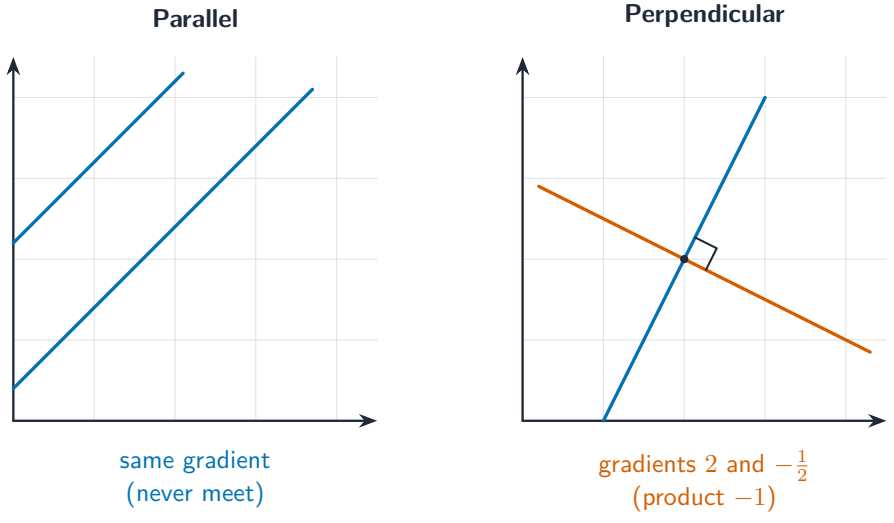
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Coordinate geometry in the (x, y) plane

A line through (x_1, y_1) with gradient m has equation $y - y_1 = m(x - x_1)$; the form $ax + by + c = 0$ hides the gradient in $-\dfrac{a}{b}$. Two points give the gradient: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.



A city street grid: every point is fixed by two coordinates, and straight roads are straight lines



The non-vertical lines shown have equal gradients when parallel and gradient product -1 when perpendicular.

- Distinct non-vertical **parallel** 平行 lines share their gradient: $m_1 = m_2$; vertical parallel lines have undefined gradients.

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- **Perpendicular** 垂直 lines with finite gradients satisfy $m_1 m_2 = -1$. A horizontal line and a vertical line are also perpendicular.

> **Worked example – perpendicular through the origin** (adapted from Edexcel IAL WMA11 June 2025 Q5). The line l_1 has equation $x - 2y + 25 = 0$; l_2 passes through the origin and is perpendicular to l_1.

(a) Equation of l_2.

- Known: rearrange l_1: $2y = x + 25$, so $y = \frac{1}{2}x + \frac{25}{2}$ and $m_1 = \frac{1}{2}$.
- Why: perpendicular gradients multiply to -1, so $m_2 = -1 \div \frac{1}{2} = -2$.

$$y - 0 = -2(x - 0) \implies y = -2x$$

(b) l_1 and l_2 intersect at P.

- Substitute $y = -2x$ into $x - 2y + 25 = 0$:

$$x - 2(-2x) + 25 = 0 \implies 5x + 25 = 0 \implies x = -5, \quad y = -2(-5) = 10$$

- $P = (-5, 10)$. Check in l_1: $-5 - 20 + 25 = 0$ ✓

(c) Shortest distance from l_1 to the origin, as a simplified surd.

- The shortest distance is $|OP|$ along the perpendicular:

$$|OP| = \sqrt{(-5-0)^2 + (10-0)^2} = \sqrt{25 + 100} = \sqrt{125} = 5\sqrt{5}$$

- Check: $\sqrt{125} = \sqrt{25 \times 5} = 5\sqrt{5} \approx 11.18$ ✓

Trigonometry

■ The sine rule, cosine rule and area of a triangle

In triangle ABC with sides a, b, c opposite angles A, B, C:



A Ferris wheel: a seat's height rises and falls like a sine curve as the wheel turns

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$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \qquad a^2=b^2+c^2-2bc\cos A \qquad \text{area}=\tfrac{1}{2}ab\sin C$$

Use the sine rule for pairs (side, opposite angle); the cosine rule when no pair is opposite. The sine rule has an **ambiguous case** 二义情形: $\sin\theta=\sin(180°-\theta)$, so an angle and its supplement can both fit – check which the diagram allows.

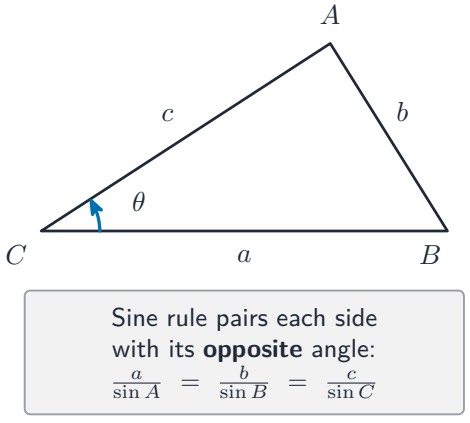
Radian measure, arc length and sector area

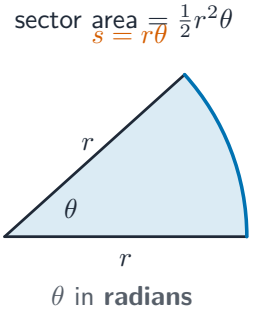
One **radian** is the angle subtended by an arc equal in length to the radius; $180°=\pi$ radians. With θ in radians:

$$\text{arc length } s=r\theta \qquad \text{sector area}=\tfrac{1}{2}r^2\theta$$

The **arc length** 弧长 is $s=r\theta$ and the **sector area** 扇形面积 is $\frac{1}{2}r^2\theta$.

A **major sector** is the larger share of the circle: angle $2\pi-\theta$ when the minor angle is θ.





Quote the formula first – the method mark follows the formula.

> **Worked example – a helipad from a major sector and a triangle** (adapted from Edexcel IAL WMA11 June 2025 Q6). A landing pad is the major sector AOB of a circle centre O joined to triangle AOC: $AO=OB=15$ m, $BC=2$ m, CBO is a straight line, and angle $ACO=0.6$ radians.

(a) Show angle $COA=1.847$ radians to 3 decimal places.

- Known: $OC=OB+BC=15+2=17$ m; in triangle AOC, side $AO=15$ is opposite the given angle 0.6, and side $OC=17$ is opposite angle CAO.

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- Sine rule (quote it first):

$$\frac{\sin CAO}{17} = \frac{\sin 0.6}{15} \implies \sin CAO = \frac{17\sin 0.6}{15} = 0.6399\ldots \implies CAO = 0.6944\ldots$$

$$COA = \pi - 0.6 - 0.6944\ldots = 1.8471\ldots = 1.847 \text{ (3 d.p.)}$$

- Check: angles sum to π: $0.6 + 0.6944 + 1.8471 \approx 3.1416$ ✓

(b) Total area of the pad.

- Major sector angle: $2\pi - 1.847 = 4.436$ radians.

$$\text{sector} = \tfrac{1}{2}r^2\theta = \tfrac{1}{2} \times 15^2 \times 4.436 = 499.1 \text{ m}^2 \text{ (awrt 499)}$$

$$\text{triangle} = \tfrac{1}{2}ab\sin C = \tfrac{1}{2} \times 15 \times 17 \times \sin 1.847 = 122.7 \text{ m}^2$$

$$\text{total} = 499.1 + 122.7 = 622 \text{ m}^2 \text{ (3 s.f.)}$$

- Check: $\sin 1.847 = \sin(\pi - 1.847) = \sin 1.2946 = 0.962$ ✓

(c) Perimeter of the pad.

- The boundary is the major arc AB plus sides AC and BC.

$$\text{arc} = r\theta = 15 \times 4.436 = 66.5 \text{ m}$$

- Cosine rule for AC (quote it first):

$$AC^2 = 15^2 + 17^2 - 2(15)(17)\cos 1.847 = 225 + 289 + 139.1 = 653.1 \implies AC = 25.6 \text{ m}$$

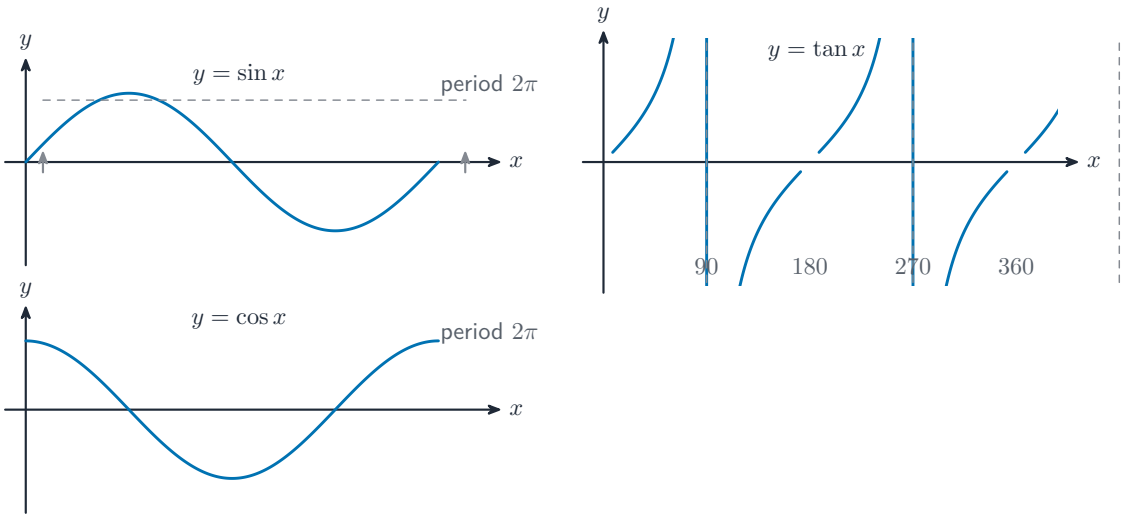
$$\text{perimeter} = 66.5 + 25.6 + 2 = 94.1 \text{ m (3 s.f.)}$$

- Check: $\cos 1.847 < 0$ (the angle is obtuse), so the $-2bc\cos A$ term adds ✓

■ Sine, cosine and tangent graphs

$y = \sin x$ and $y = \cos x$ repeat every 2π (360°); $y = \tan x$ repeats every π (180°) with asymptotes at $x = 90° + 180°n$. Know the symmetries: cos is even, $\sin(180° - x) = \sin x$, and $\tan x = \sin x / \cos x$. Curves such as $y = 3\sin x$, $y = \sin 2x$ and $y = \sin\left(x + \frac{\pi}{6}\right)$ follow from the transformation rules above. Solve equations **graphically** by counting intersections in a given interval.

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Count periods first, then solutions per period.

Worked example – counting solutions (adapted from Edexcel IAL WMA11 June 2025 Q4). For $y = 4\cos x$ with x in degrees: how many solutions does $5 + 4\cos x = 1$ have for $-720° < x \leq 720°$?

- Rearrange: $4\cos x = -4$, so $\cos x = -1$, giving $x = 180° + 360°n$.
- In $-720° < x \leq 720°$: $x = -540°, -180°, 180°, 540°$ – **4 solutions**.
- Check: $\cos(\pm 180°) = -1$ ✓, and $5 + 4(-1) = 1$ ✓. The interval is half-open, so $x = -720°$ (where $\cos = 1$) is excluded correctly.

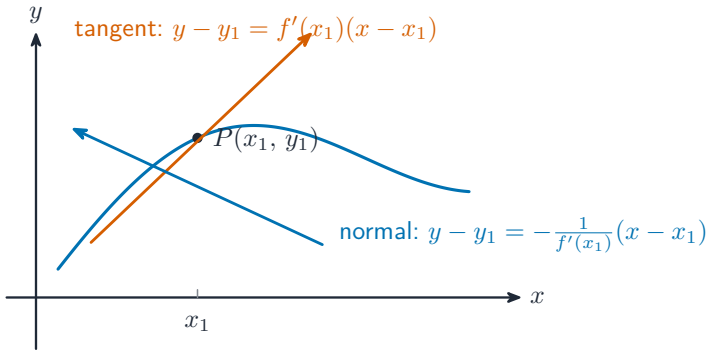
Differentiation

The **derivative** 导数 $f'(x) = \dfrac{dy}{dx}$ is the **gradient of the tangent** 切线斜率 to $y = f(x)$ at a point – the limit of the chord gradients as the two points merge. It is the **rate of change** 变化率 of y with respect to x. The **second derivative** 二阶导数 $f''(x)$ differentiates again. (The chain rule is **not** required in P1.)

Differentiate term by term with the power rule, expanding products and splitting fractions first:

$$y = (2x+5)(x-1) = 2x^2 + 3x - 5 \implies \frac{dy}{dx} = 4x + 3 \qquad y = \frac{3x^2+5}{x^2} = 3 + 5x^{-2} \implies \frac{dy}{dx} = -10x^{-3}$$

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Tangent gradient $= f'(x_1)$*; for nonzero finite tangent gradient, normal gradient* $= -1/f'(x_1)$*. A horizontal tangent has a vertical normal.*

Worked example – gradient condition on a curve (adapted from Edexcel IAL WMA11 June 2025 Q2). C: $y = 2x^{5/2} - 4x + 3$; the point P on C has x-coordinate 2^k and the gradient at P is 16. Find k.

- Differentiate (powers drop by one): $\dfrac{dy}{dx} = 2 \cdot \frac{5}{2}x^{3/2} - 4 = 5x^{3/2} - 4$.
- Set the gradient to 16 at $x = 2^k$:

$$5(2^k)^{3/2} - 4 = 16 \implies (2^k)^{3/2} = 4 = 2^2 \implies 2^{3k/2} = 2^2 \implies \tfrac{3k}{2} = 2 \implies k = \tfrac{4}{3}$$

- Check: $2^{4/3} = 2.5198\ldots$ and $5(2.5198)^{3/2} - 4 = 5(4.0) - 4 = 16$ ✓

Worked example – tangent, normal work and second derivative (adapted from Edexcel IAL WMA11 June 2025 Q8). A curve has $f'(x) = 3\sqrt{x} + kx^2$ and passes through $P(4, 12)$; the tangent at P is $y = 10x + c$.

(a) Show $k = \frac{1}{4}$ and find c.

- Known: the tangent's gradient equals $f'(4)$:

$$f'(4) = 3\sqrt{4} + 16k = 6 + 16k = 10 \implies k = \tfrac{1}{4}$$

- The tangent passes through $(4, 12)$: $12 = 10(4) + c \implies c = -28$.
- Check: $f'(4) = 6 + 16 \cdot \frac{1}{4} = 10$ ✓ and $10(4) - 28 = 12$ ✓

(b) $f''(x)$ at P.

$$f''(x) = \tfrac{3}{2}x^{-1/2} + 2kx = \tfrac{3}{2\sqrt{x}} + \tfrac{x}{2} \implies f''(4) = \tfrac{3}{4} + 2 = \tfrac{11}{4}$$

- Check: $\frac{3}{2\sqrt{4}} = \frac{3}{4}$ ✓

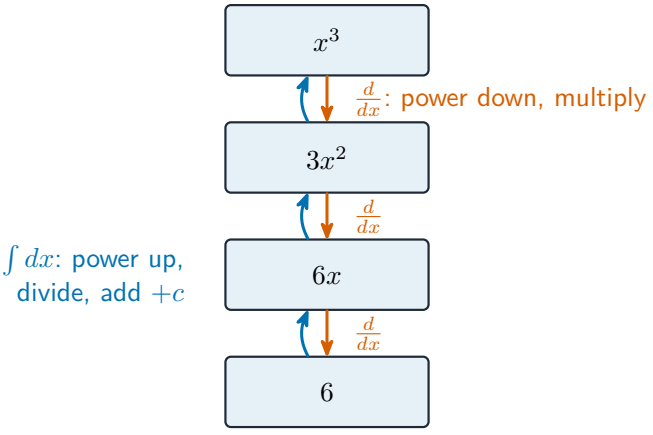
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Integration

Integration 积分 reverses differentiation: if $\dfrac{dy}{dx} = x^n$ $(n \neq -1)$, then

$$y = \int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

The **constant of integration** 积分常数 c is required – every vertical shift shares the same derivative. Raise each power by one, divide by the new power, and integrate term by term after expanding. Given $f'(x)$ **and one point** on the curve, find c and recover $y = f(x)$ exactly.



Each integral climbs one rung and picks up an arbitrary constant: $\int 6 \, dx = 6x + c$, not just $6x$.

Differentiation slides down; integration climbs back up, plus c.

Worked example – expand before integrating (adapted from Edexcel IAL WMA11 June 2025 Q3). Find $\displaystyle\int \frac{(x+3)^2}{3\sqrt{x}} \, dx$.

- Expand the numerator: $(x+3)^2 = x^2 + 6x + 9$; the denominator is $3x^{1/2}$.
- Divide term by term (powers of x subtract):

$$\frac{x^2}{3x^{1/2}} = \tfrac{1}{3}x^{3/2} \qquad \frac{6x}{3x^{1/2}} = 2x^{1/2} \qquad \frac{9}{3x^{1/2}} = 3x^{-1/2}$$

- Integrate each term (power up one, divide by the new power):

$$\int \left(\tfrac{1}{3}x^{3/2} + 2x^{1/2} + 3x^{-1/2}\right) dx = \tfrac{1}{3} \cdot \tfrac{2}{5}x^{5/2} + 2 \cdot \tfrac{2}{3}x^{3/2} + 3 \cdot 2x^{1/2} + c$$

$$= \frac{2}{15}x^{5/2} + \frac{4}{3}x^{3/2} + 6\sqrt{x} + c$$

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- Check by differentiating: $\frac{2}{15}\cdot\frac{5}{2}x^{3/2}=\frac{1}{3}x^{3/2}$ ✓, $\frac{4}{3}\cdot\frac{3}{2}=2$ ✓, $6\cdot\frac{1}{2}=3$ ✓ – the integrand returns exactly.

Worked example – recover $f(x)$ from $f'(x)$ and a point (adapted from Edexcel IAL WMA11 June 2025 Q8, continued). With $k=\frac{1}{4}$, so $f'(x)=3x^{1/2}+\frac{1}{4}x^2$, and $P(4,12)$ on the curve:

$$f(x)=\int\left(3x^{1/2}+\tfrac{1}{4}x^2\right)dx=3\cdot\tfrac{2}{3}x^{3/2}+\tfrac{1}{4}\cdot\tfrac{1}{3}x^3+d=2x^{3/2}+\frac{x^3}{12}+d$$

- Substitute P: $2(4)^{3/2}+\frac{64}{12}+d=12$, and $4^{3/2}=8$:

$$16+\tfrac{16}{3}+d=12\implies d=-\tfrac{28}{3}\implies f(x)=2x^{3/2}+\frac{x^3}{12}-\frac{28}{3}$$

- Check: $f(4)=16+\frac{16}{3}-\frac{28}{3}=16-4=12$ ✓, and differentiating the answer gives $f'(x)=3x^{1/2}+\frac{x^2}{4}$, so $f'(4)=6+4=10$ ✓ (the tangent gradient from before).

Checklist before you call this unit done

- Manipulate $a^{m/n}$, negative powers and surds exactly, and rationalise denominators.
- Read b^2-4ac three ways, complete the square, and solve hidden quadratics by substitution.
- Solve linear–quadratic simultaneous equations by substitution and interpret them as intersections.
- Solve and sketch linear and quadratic inequalities with correct dotted/solid boundaries.
- Sketch cubics, reciprocals (with asymptotes) and transformed curves, naming direction, type and scale factor of each single transformation.
- Write line equations in both forms; use $m_1m_2=-1$ for perpendiculars.
- Quote sine rule, cosine rule and $\frac{1}{2}ab\sin C$; handle the ambiguous case and major sectors in radians.
- Count trigonometric-graph solutions over an interval before listing them.
- Differentiate and integrate x^n term by term, after expanding; always add $+c$.
- Turn $f'(x)$ plus one point into $f(x)$, and check by substituting the point back.

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2 · Pure Mathematics 2

Pearson Edexcel · International A-Level · Mathematics

Pure Mathematics 2: building on the P1 toolkit

Pure Mathematics 2 (P2) is the second compulsory unit of International AS Mathematics and International AS Pure Mathematics, and of both International A Level routes. This reference covers every P2 topic in the Edexcel IAL specification (Issue 3, April 2019): proof, the factor and remainder theorems, circle geometry, sequences and series, exponentials and logarithms, trigonometric identities and equations, stationary points, and integration including areas and the trapezium rule.

- The P2 paper (WMA12/01) lasts **1 hour 30 minutes** with **75 marks**. A knowledge of P1 and its formulae is assumed and **may be tested** on the P2 paper.
- The same calculator and yellow *Mathematical Formulae and Statistical Tables* rules apply as for P1. The formulae below do **not** appear in the booklet – memorise them.
- "Show all stages of your working" questions reject calculator-only answers; give inexact answers to 3 significant figures unless told otherwise.

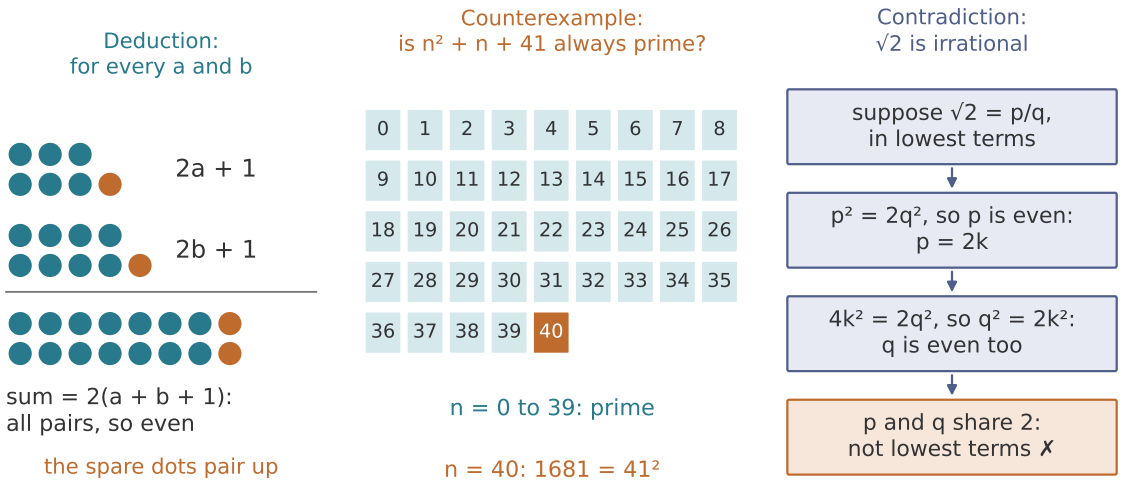
Formulae you must know for P2 (from the specification's assessment information):

Formula	Use
$\log_a x + \log_a y \equiv \log_a(xy)$, $\log_a x - \log_a y \equiv \log_a \dfrac{x}{y}$, $k \log_a x \equiv \log_a(x^k)$	laws of logarithms 对数运算法则
$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$, $\sin^2\theta + \cos^2\theta \equiv 1$	trigonometric identities 三角恒等式
area under a curve $= \displaystyle\int_a^b y\,dx \quad (y \geq 0)$	definite integral as an area

Proof

A **proof** 证明 proceeds from given assumptions through logical steps to a conclusion. P2 uses three methods:

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Deduction proves every case at once; one counterexample ($n = 40$) disproves an "always" claim; contradiction assumes the opposite and breaks it

- **Proof by exhaustion** 穷举法: split the possibilities into cases and check every case.
- **Disproof by counterexample** 反例法: one example that breaks the claim disproves it.
- Algebraic proof: start from what is given, manipulate identities (not equations to solve), and end with the target statement.

Worked example – exhaustion (adapted from Edexcel IAL WMA12 June 2025 Q10). Prove that for all positive integers m that are not multiples of 3, the value of $m^2 + 3m + 2$ is always a multiple of 3.

- Factorise first: $m^2 + 3m + 2 = (m + 1)(m + 2)$.
- A positive integer not divisible by 3 is either $m = 3q + 1$ or $m = 3q + 2$ – those are all the cases, so exhausting them proves the claim.
- Case $m = 3q + 1$: $(m + 1)(m + 2) = (3q + 2)(3q + 3) = 3(3q + 2)(q + 1)$ – a multiple of 3.
- Case $m = 3q + 2$: $(m + 1)(m + 2) = (3q + 3)(3q + 4) = 3(q + 1)(3q + 4)$ – a multiple of 3.
- Both cases give a multiple of 3, so $3 \mid m^2 + 3m + 2$ whenever $3 \nmid m$. ■
- Check: $m = 1$: $6 = 3 \times 2$ ✓; $m = 2$: $12 = 3 \times 4$ ✓; $m = 4$: 30 ✓; $m = 5$: 42 ✓.

Worked example – counterexample. Disprove: "$n^2 - n + 1$ is prime for all integers $n \geq 2$."

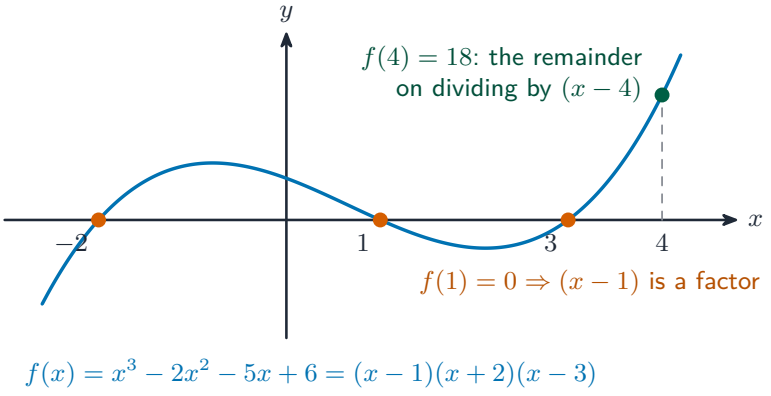
- One counterexample is enough. Try $n = 5$: $25 - 5 + 1 = 21 = 3 \times 7$, which is not prime.

- The claim is false. Check $n = 2, 3, 4$ give the primes $3, 7, 13$ – the pattern holds at first, which is exactly why a counterexample, not more examples, settles the question.

Algebra: the factor and remainder theorems

For a polynomial $f(x)$:

- **Factor theorem** 因式定理: $(x - b)$ is a factor of $f(x)$ exactly when $f(b) = 0$. For divisors of the form $(ax - b)$: if $f\left(\frac{b}{a}\right) = 0$ then $(ax - b)$ is a factor.
- **Remainder theorem** 余项定理: dividing $f(x)$ by $(ax - b)$ leaves remainder $f\left(\frac{b}{a}\right)$.
- **Algebraic division** 代数除法 by $(ax \pm b)$ gives a **quotient** 商 and a **remainder**; the degree drops by one: $f(x) = (ax - b) \times (\text{quotient}) + \text{remainder}$.



Roots are factors; the height at $x = b$ is the remainder on dividing by $(x - b)$

> **Worked example – both theorems at once** (adapted from Edexcel IAL WMA12 June 2025 Q4). $f(x) = ax^3 + bx^2 + 5x - 3$, where $(x + 3)$ is a factor of $f(x)$ and dividing $f(x)$ by $(2x - 1)$ leaves remainder 7.

(a) Show that $b = 3a + 2$.

- Factor theorem: $f(-3) = 0$:

$$a(-27) + b(9) + 5(-3) - 3 = 0 \implies -27a + 9b - 18 = 0 \implies b = 3a + 2$$

(b) Find a and b.

- Remainder theorem with $(2x - 1)$: $f\left(\frac{1}{2}\right) = 7$:

$$\frac{a}{8} + \frac{b}{4} + \frac{5}{2} - 3 = 7 \implies \frac{a}{8} + \frac{b}{4} = 7.5 \implies a + 2b = 60$$

- Substitute $b = 3a + 2$: $a + 6a + 4 = 60$, so $a = 8$ and $b = 26$.

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- Check: $f(x) = 8x^3 + 26x^2 + 5x - 3$; $f(-3) = -216 + 234 - 15 - 3 = 0$ ✓ and $f(\frac{1}{2}) = 1 + 6.5 + 2.5 - 3 = 7$ ✓.

(c) Quotient and remainder when $f(x)$ is divided by $(x - 2)$.

- Divide $8x^3 + 26x^2 + 5x - 3$ by $(x - 2)$: quotient $8x^2 + 42x + 89$, remainder 175.
- Check by rebuilding: $(x - 2)(8x^2 + 42x + 89) + 175 = 8x^3 + 42x^2 + 89x - 16x^2 - 84x - 178 + 175 = 8x^3 + 26x^2 + 5x - 3$ ✓. The remainder also equals $f(2) = 64 + 104 + 10 - 3 = 175$ ✓.

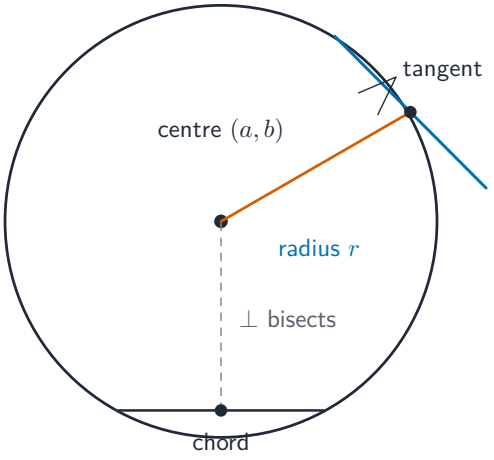
Coordinate geometry of the circle

A circle centre (a, b) radius r has equation

$$(x - a)^2 + (y - b)^2 = r^2$$

Given the expanded form $x^2 + y^2 + fx + gy + c = 0$, complete the square to recover centre $\left(-\frac{f}{2}, -\frac{g}{2}\right)$ and $r^2 = \frac{f^2+g^2}{4} - c$. Three circle properties are examined:

- the angle in a semicircle is a right angle;
- the perpendicular from the centre to a **chord** 弦 bisects the chord;
- a **tangent** 切线 is perpendicular to the radius at the point of contact.



Tangent ⊥ radius at the contact point; the perpendicular from the centre to a chord **bisects** it; the angle in a semicircle is 90.

Radius ⊥ tangent; centre-to-chord perpendicular bisects the chord.

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Worked example – diameter endpoints (adapted from Edexcel IAL WMA12 June 2025 Q2). The line joining $(-2, 5)$ and $(4, 15)$ is a diameter of circle C.

(a) Equation of C.

- The centre is the midpoint: $\left(\frac{-2+4}{2}, \frac{5+15}{2}\right) = (1, 10)$.
- The diameter squared: $d^2 = (4-(-2))^2 + (15-5)^2 = 36 + 100 = 136$, so $r^2 = 136 \div 4 = 34$.

$$(x-1)^2 + (y-10)^2 = 34$$

- Check: both endpoints satisfy the equation: $(-3)^2+(-5)^2 = 34$ ✓ and $3^2+5^2 = 34$ ✓.

(b) Exact coordinates of the point on C nearest the x-axis.

- The nearest point lies on the vertical radius: same x as the centre, r below it:

$$(1,\ 10-\sqrt{34})$$

- Check: $(1-1)^2 + (10-\sqrt{34}-10)^2 = 34$ ✓, and no smaller y is possible since the centre sits $10 > \sqrt{34}$ above the axis.

Sequences and series

A **sequence** 数列 may be given by a formula for the nth term or by a relation $x_{n+1} = f(x_n)$. Describe it as **increasing** 递增, **decreasing** 递减 or **periodic** 周期 (from its terms, not its formula).



Romanesco broccoli: each bud repeats the whole at a smaller scale, a geometric pattern

Arithmetic sequences 等差数列: constant difference d.

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$$a_n = a + (n-1)d \qquad S_n = \frac{n}{2}\big(2a + (n-1)d\big) \qquad \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

The proof of the sum formula (write S_n forwards and backwards, add) should be known, as should Σ notation.

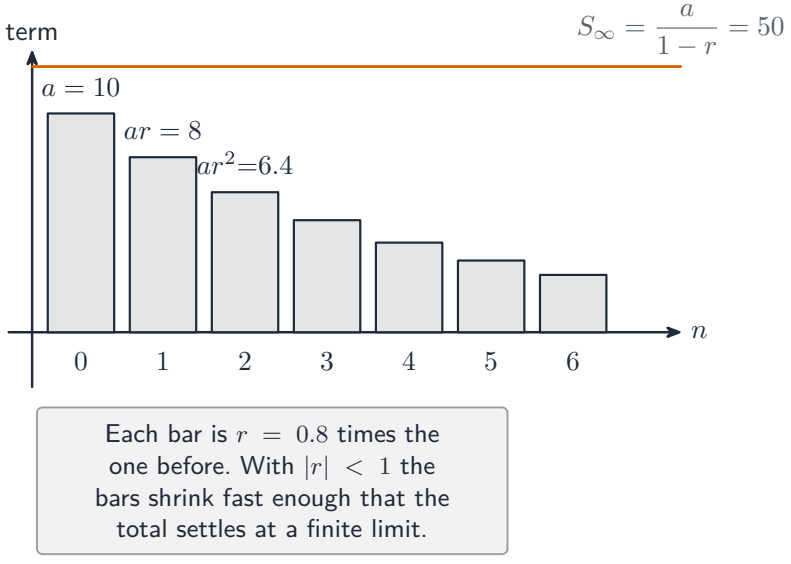
Geometric sequences 等比数列: constant ratio r.

$$a_n = ar^{n-1} \qquad S_n = \frac{a(1-r^n)}{1-r} \qquad S_\infty = \frac{a}{1-r}\ (|r|<1)$$

The sum formula's proof (telescoping $S_n - rS_n$) should be known. Use **logarithms** to solve for n when a term value is given.

Binomial expansion 二项展开式 for a positive integer n:

$$(a+bx)^n = a^n + \binom{n}{1}a^{n-1}(bx) + \binom{n}{2}a^{n-2}(bx)^2 + \cdots + (bx)^n, \qquad \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$



$|r| < 1$ means the terms shrink; the sum settles at S_∞.

Worked example – geometric series with logs (adapted from Edexcel IAL WMA12 June 2025 Q7). A geometric series begins $10 + 8 + 6.4 + \cdots$

(a) Sum to infinity.

- $a = 10$, $r = 8/10 = 0.8$, and $|r| < 1$:

$$S_\infty = \frac{a}{1-r} = \frac{10}{1-0.8} = 50$$

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(b) The kth term is less than 0.0005 – find the smallest k, by algebra.

- The kth term is $10(0.8)^{k-1}$:

$$10(0.8)^{k-1} < 0.0005 \implies (0.8)^{k-1} < 0.00005$$

- Take logs (the log of a number below 1 is negative – keep the inequality direction when dividing by it):

$$(k-1)\log 0.8 < \log 0.00005 \implies k - 1 > \frac{\log 0.00005}{\log 0.8} = 44.38\ldots$$

- So $k > 45.38$ and the smallest integer is $k = 46$.
- Check: $10(0.8)^{45} = 0.000436\ldots < 0.0005$ ✓ but $10(0.8)^{44} = 0.000544\ldots > 0.0005$ ✓.

Worked example – maximum arithmetic sum (adapted from the same question). An arithmetic series begins $850 + 843 + 836 + \cdots$ ($d = -7$). Its greatest partial sum:

- Terms stay positive until $u_n = 850 - 7(n-1) > 0$, so $n \leq 122.4$; the 122nd term is $850 - 7(121) = 3 > 0$ and the 123rd is $-4 < 0$.

$$S_{122} = \frac{122}{2}\big(2(850) + 121(-7)\big) = 61(1700 - 847) = 61 \times 853 = 52\,033$$

- Check: adding a negative 123rd term would reduce the sum, so S_{122} is the maximum ✓.

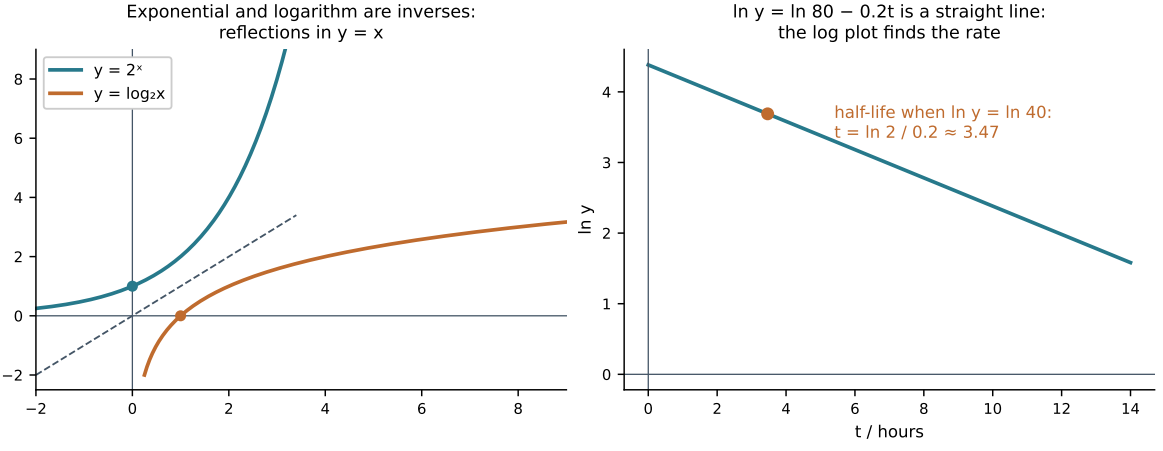
Exponentials and logarithms

$y = a^x$ ($a > 0$, $a \neq 1$) passes through $(0, 1)$ and grows without bound for $a > 1$. The **logarithm** 对数 $\log_a x$ inverts it: $\log_a x = y$ means $a^y = x$. Know:

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A seismograph: earthquake magnitude is a logarithmic scale, so each step up is ten times the shaking



Exponential and log are inverse functions; ln y against t is a straight line, and the half-life is ln 2 / 0.2 ≈ 3.47 hours

$$\log_a x + \log_a y \equiv \log_a(xy) \qquad \log_a x - \log_a y \equiv \log_a \frac{x}{y} \qquad k\log_a x \equiv \log_a(x^k) \qquad \log_a a = 1$$

and $\log_a \frac{1}{x} = -\log_a x$. Solve $a^x = b$ with $x = \log_a b$ (a change of base may be used). Always check candidates against the **domain** 定义域: logs need positive arguments.

> **Worked example – a log equation** (adapted from Edexcel IAL WMA12 June 2025 Q6). Solve $2\log_2(4-x) = 3 + \log_2\left(\dfrac{x+11}{2}\right)$.

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- Bring everything to one log per side:

$$\log_2(4-x)^2 = \log_2 8 + \log_2\left(\frac{x+11}{2}\right) = \log_2\big(4(x+11)\big)$$

- Equal logs give equal arguments:

$$(4-x)^2 = 4(x+11) \implies x^2 - 8x + 16 = 4x + 44 \implies x^2 - 12x - 28 = 0$$

$$(x-14)(x+2) = 0 \implies x = 14 \text{ or } x = -2$$

- Domain check: $4 - x > 0$ needs $x < 4$, so $x = 14$ is rejected. $x = -2$.
- Check: LHS $= 2\log_2 6$; RHS $= 3 + \log_2(4.5) = \log_2(8 \times 4.5) = \log_2 36 = 2\log_2 6$ ✓.

Worked example – exact intersection in $\log_3$ form (same question). The curves $y = 3^{2x+1}$ and $y = 6 \times 3^x$ meet at P; write P as $(\log_3 a, b)$.

- Set equal and substitute $u = 3^x$:

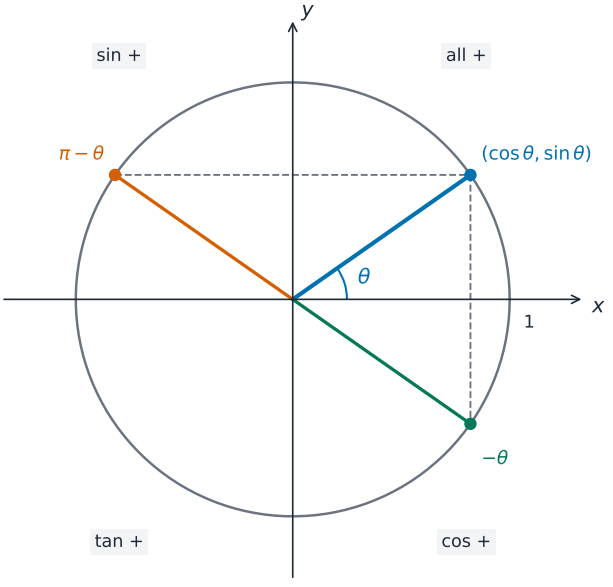
$$3 \cdot 3^{2x} = 6 \cdot 3^x \implies 3u^2 = 6u \implies u = 2 \implies 3^x = 2 \implies x = \log_3 2$$

$$y = 6 \times 3^x = 6 \times 2 = 12$$

- $P = (\log_3 2,\ 12)$, so $a = 2$, $b = 12$. Check: $3^{2\log_3 2 + 1} = 3 \cdot 3^{2\log_3 2} = 3 \cdot 4 = 12$ ✓.

Trigonometric identities and equations

Two identities convert between sine, cosine and tangent:



Reflections in the unit circle give the symmetry rules

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$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta \equiv 1$$

Solve **trigonometric equations** 三角方程 in a given interval by reducing to one function (often a hidden quadratic), then reading the second solution from the graph's symmetry. State the interval, work in its units (degrees or radians), and reject values outside.

Worked example – radians and exact fractions (adapted from Edexcel IAL WMA12 June 2025 Q8). Solve $3\tan\left(2x + \dfrac{\pi}{5}\right) = \sqrt{3}$ for $0 \le x < \pi$.

- Divide by 3: $\tan\left(2x + \frac{\pi}{5}\right) = \dfrac{1}{\sqrt{3}}$, and $\tan\frac{\pi}{6} = \dfrac{1}{\sqrt{3}}$:

$$2x + \frac{\pi}{5} = \frac{\pi}{6} + k\pi \implies 2x = -\frac{\pi}{30} + k\pi \implies x = -\frac{\pi}{60} + \frac{k\pi}{2}$$

- In $0 \le x < \pi$: $x = \dfrac{29\pi}{60}$ (k = 1) and $x = \dfrac{59\pi}{60}$ (k = 2).
- Check: at $x = \frac{29\pi}{60}$: $2x + \frac{\pi}{5} = \frac{7\pi}{6}$ and $\tan\frac{7\pi}{6} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ ✓.

Worked example – hidden quadratic in cosine (same question). Solve $5\sin\theta\tan\theta = \cos\theta + 4$ for $0 \le \theta < 360°$.

- Replace $\tan\theta = \sin\theta/\cos\theta$ and multiply through by $\cos\theta$:

$$\frac{5\sin^2\theta}{\cos\theta} = \cos\theta + 4 \implies 5\sin^2\theta = \cos^2\theta + 4\cos\theta$$

- Use $\sin^2\theta = 1 - \cos^2\theta$:

$$5(1 - \cos^2\theta) = \cos^2\theta + 4\cos\theta \implies 6\cos^2\theta + 4\cos\theta - 5 = 0$$

- Quadratic formula in $\cos\theta$:

$$\cos\theta = \frac{-4 \pm \sqrt{16 + 120}}{12} = \frac{-2 \pm \sqrt{34}}{6}$$

- The $-$ root is below -1 (reject); $\cos\theta = 0.6385$ gives $\theta = 50.3°$ and, from cosine's symmetry, $\theta = 360° - 50.3° = 309.7°$.
- Check: $\cos 50.3° = 0.638$ ✓; both angles lie in the interval ✓.

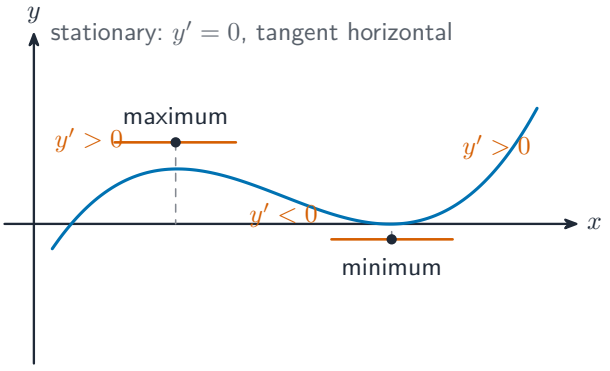
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Differentiation: stationary points

Stationary points 驻点 have $\dfrac{dy}{dx} = 0$. Classify with the sign of the derivative:

- derivative changes $+$ to $-$: **maximum** 极大值;
- derivative changes $-$ to $+$: **minimum** 极小值;
- the second derivative helps: $f''(x) > 0$ at the point confirms a minimum, $f''(x) < 0$ a maximum.

A function is **increasing** where $f'(x) > 0$ and **decreasing** where $f'(x) < 0$. Practical problems – least material, least cost – translate into minimising an expression in one variable.



Stationary means the tangent is horizontal; the sign change classifies it.

Worked example – a modelling minimum (adapted from Edexcel IAL WMA12 June 2025 Q9). An open trough has right-triangle ends with legs $3x$ and $2x$ (metres), rectangular sides $3x \times L$ and $2x \times L$, and capacity 12 m^3. The metal area is S m^2.

(a) Show $S = 6x^2 + \dfrac{20}{x}$.

- Volume: cross-section $\times$ length $= \frac{1}{2}(3x)(2x) \times L = 3x^2L = 12$, so $L = \dfrac{4}{x^2}$.
- Surface: two triangles plus the two rectangles:

$$S = 2 \times \tfrac{1}{2}(3x)(2x) + 3xL + 2xL = 6x^2 + 5x \cdot \frac{4}{x^2} = 6x^2 + \frac{20}{x}$$

(b) Minimum S by calculus.

$$\frac{dS}{dx} = 12x - \frac{20}{x^2} = 0 \implies 12x^3 = 20 \implies x^3 = \frac{5}{3} \implies x = 1.1856\ldots$$

$$S = 6(1.1856)^2 + \frac{20}{1.1856} = 8.43 + 16.87 = 25.3 \text{ m}^2 \text{ (1 d.p.)}$$

(c) Justify the minimum.

$$\frac{d^2S}{dx^2} = 12 + \frac{40}{x^3} > 0 \quad \text{for all } x > 0$$

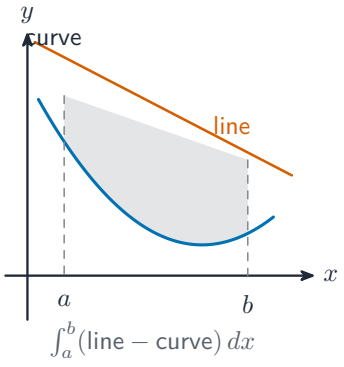
- Positive second derivative everywhere on the domain: the stationary point is a minimum ✓.
- Check: at $x = 1.1856$, $\frac{d^2S}{dx^2} = 12 + 40/(5/3) = 12 + 24 = 36 > 0$ ✓.

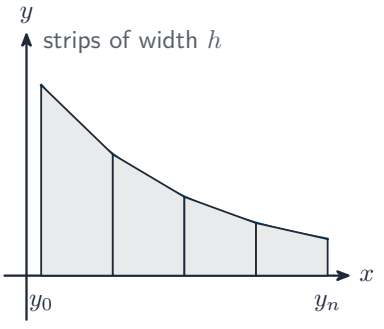
Integration: areas and the trapezium rule

Evaluate **definite integrals** 定积分 with the fundamental theorem:

$$\int_a^b y\,dx = F(b) - F(a) \quad \text{where } F'(x) = y$$

The definite integral is the **area under the curve** 曲线下面积 for $y \geq 0$. For a region bounded by a curve and a line, integrate (upper − lower) between their intersections, splitting wherever their order changes. Check which graph is above the other.





Area between uses $\int$(upper − lower), splitting at crossings; the trapezium rule joins neighbouring values with straight strips.

Worked example – area between curve and line (adapted from Edexcel IAL WMA12 June 2025 Q3). C: $y = 3x + \dfrac{16}{x^2} - 8$ $(x > 0)$ with $P(1, 11)$ and $Q(4, 5)$ on C; find the area between C and the line segment PQ.

- Line PQ: gradient $\dfrac{5-11}{4-1} = -2$, so $y - 11 = -2(x-1) \implies y = -2x + 13$.
- Check Q: $-8 + 13 = 5$ ✓.
- Area under the line: $\displaystyle\int_1^4 (-2x + 13)\,dx = \left[-x^2 + 13x\right]_1^4 = 36 - 12 = 24$.

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- Area under the curve: $\int_1^4 \left(3x + 16x^{-2} - 8\right) dx = \left[\frac{3}{2}x^2 - \frac{16}{x} - 8x\right]_1^4 = (24 - 4 - 32) - (\frac{3}{2} - 16 - 8) = 10.5$.
- Region area: $24 - 10.5 = \mathbf{13.5}$ square units.
- Check: the line is above the curve throughout (at $x = 2$: line 9, curve $3(2)+4-8 = 2$), so subtracting curve from line is the right way round ✓.

The trapezium rule 梯形法则 approximates an area with n strips of width h:

$$\int_a^b y\,dx \approx h\left[\tfrac{1}{2}(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1}\right] \quad \text{or} \quad \frac{h}{2}\left[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})\right]$$

More strips (or a straighter curve) improve the estimate; a curve that bends upwards (**convex**, U-shaped like x^2) leaves every chord **above** it, so the rule **overestimates**, while a curve that bends downwards (**concave**, like $\sqrt{x}$ or a circle arc) leaves chords **below** it, so the rule **underestimates**.

Worked example – trapezium rule with a table. Estimate $\int_1^3 \frac{1}{x}\,dx$ with 4 strips.

- Four strips over $[1, 3]$ give $h = 0.5$ and five values of $y = \dfrac{1}{x}$: at $x = 1,\ 1.5,\ 2,\ 2.5,\ 3$ the values are $1,\ 0.6667,\ 0.5,\ 0.4,\ 0.3333$.

$$\int_1^3 \frac{1}{x}\,dx \approx \frac{0.5}{2}\left[1 + 0.3333 + 2(0.6667 + 0.5 + 0.4)\right] = 0.25 \times 4.4667 = 1.12$$

- The true value is $\ln 3 = 1.099$ (3 s.f.). The estimate is an **overestimate**: $y = 1/x$ is convex (it bends upwards), so every straight strip sits above the curve ✓.

Checklist before you call this unit done

- Structure a proof; use exhaustion over complete cases; disprove with one counterexample.
- Apply the factor and remainder theorems, including $(ax - b)$ divisors, and carry out algebraic division with quotient and remainder.
- Find a circle's centre and radius from either form; use the three circle properties.
- Use a_n, S_n and S_∞ for arithmetic and geometric series (both sum-formula proofs), Σ notation, and logs to find n.
- Expand $(a + bx)^n$ with binomial coefficients, and match given coefficients.
- Solve log equations with the three laws, checking domains; give intersections exactly as $(\log_a b, c)$ when asked.
- Reduce trigonometric equations with $\tan = \sin / \cos$ and $\sin^2 + \cos^2 = 1$; solve in degrees or radians as given, listing all interval solutions.

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- Find and classify stationary points; justify with a sign change or second derivative; set up modelling minima in one variable.
- Evaluate definite integrals; find areas under curves and between curve and line; apply the trapezium rule from a table and state the direction of its error.

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3 · Pure Mathematics 3

Pearson Edexcel · International A-Level · Mathematics

Pure Mathematics 3: functions, identities and calculus at speed

Pure Mathematics 3 (P3) extends P1 and P2: rational functions and inverses, the modulus, combined transformations, reciprocal trigonometric functions and harmonic form, log-graph model fitting, the full differentiation rules, integration by recognition, and numerical methods. P1 and P2 knowledge is assumed and may be tested on the WMA13/01 paper (1 hour 30 minutes, 75 marks; same calculator and formula-booklet rules).

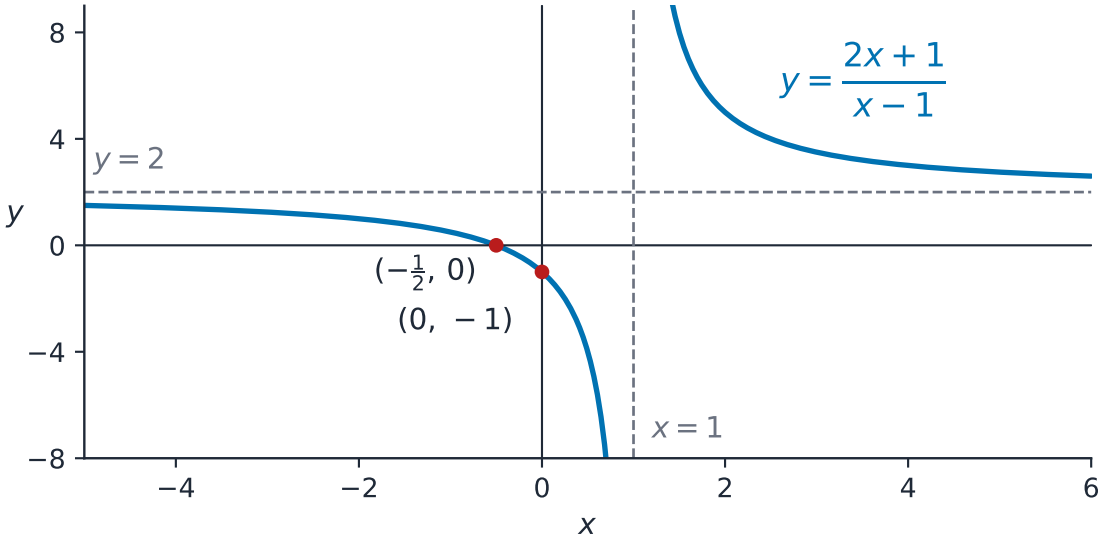
Formulae you must know for P3 (not in the booklet):

Family	Formulae
compound angle	$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$; $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$; $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
double angle	$\sin 2A \equiv 2 \sin A \cos A$; $\cos 2A \equiv \cos^2 A - \sin^2 A$; $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$
reciprocal identities	$\sec^2 A \equiv 1 + \tan^2 A$; $\operatorname{cosec}^2 A \equiv 1 + \cot^2 A$
derivatives	$\sin kx \to k \cos kx$; $\cos kx \to -k \sin kx$; $\tan kx \to k \sec^2 kx$; $e^{kx} \to k e^{kx}$; $\ln x \to \frac{1}{x}$; $a^x \to a^x \ln a$
integrals	$\int \cos kx = \frac{1}{k} \sin kx$; $\int \sin kx = -\frac{1}{k} \cos kx$; $\int e^{kx} = \frac{1}{k} e^{kx}$; $\int \frac{1}{x} = \ln\lvert x \rvert$

Algebra: rational expressions and functions

Simplify rational expressions 有理式 by factorising and cancelling; **algebraic division** serves denominators up to quadratics. Only cancel factors, never terms.

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Asymptotes first, then intercepts, then the two branches

A **function** 函数 is a one-one or many-one mapping; its **domain** 定义域 is the allowed inputs and its **range** 值域 the outputs. The **composition** 复合 fg means "do g first, then f". The **inverse** 反函数 f^{-1} undoes f (it exists where f is one-one), and $f^{-1}f(x) = ff^{-1}(x) = x$; the graph of f^{-1} reflects the graph of f in $y = x$.

Worked example – inverse and composition (adapted from Edexcel IAL WMA13 June 2025 Q1). $f(x) = \dfrac{2x}{3x+1}$, $x \geq 0$.

(a) Range of f.

- At $x = 0$: $f(0) = 0$; as $x \to \infty$: $f(x) \to \frac{2}{3}$ from below, and f is increasing (larger x gives larger $f(x)$).

$$0 \leq f(x) < \tfrac{2}{3}$$

(b) Find $f^{-1}(x)$.

- Let $y = \dfrac{2x}{3x+1}$ and solve for x:

$$y(3x+1) = 2x \implies 3xy + y = 2x \implies x(3y-2) = -y \implies x = \frac{y}{2-3y}$$

$$f^{-1}(x) = \frac{x}{2-3x}$$

- Check: $f\left(\frac{x}{2-3x}\right) = \dfrac{2x/(2-3x)}{3x/(2-3x)+1} = \dfrac{2x}{3x+2-3x} = x$ ✓.

(c) Solve $f^{-1}(x) = f(x)$.

$$\frac{x}{2-3x} = \frac{2x}{3x+1} \implies x(3x+1) = 2x(2-3x) \implies x\big(3x+1-4+6x\big) = 0$$

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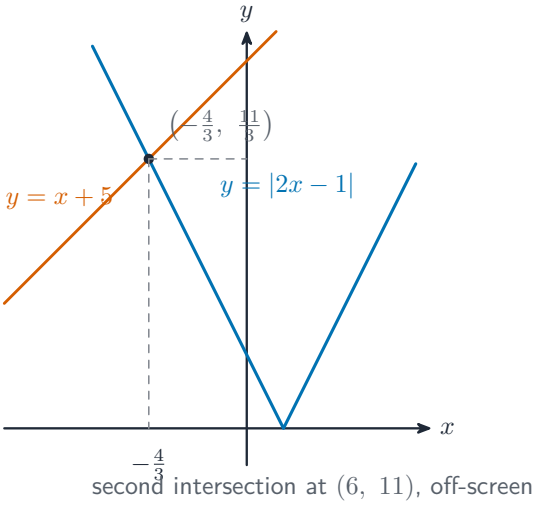
$$x(9x-3)=0 \implies x=0 \text{ or } x=\tfrac{1}{3}$$

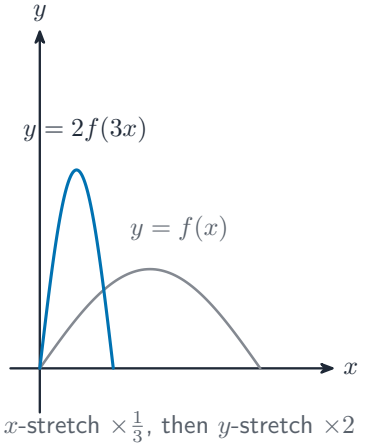
- Check: $f(\tfrac{1}{3})=\frac{2/3}{2}=\tfrac{1}{3}=f^{-1}(\tfrac{1}{3})$ ✓.

The modulus function and combined transformations

The **modulus** 绝对值 $|x|$ is x for $x\geq 0$ and $-x$ for $x<0$: $y=|ax+b|$ is the line $y=ax+b$ with its negative part reflected above the axis. Given $y=f(x)$, sketch $y=|f(x)|$ (reflect any below-axis part) and $y=f(|x|)$ (keep the right half, reflect it left). Solve modulus equations graphically or by cases, testing each solution.

Combined transformations 组合变换 apply the P1 rules together, for example $y=2f(3x)$ or $y=f(-x)+1$: stretches multiply, translations then shift; the order given in the syllabus is $y=af(x)$, $y=f(x)+a$, $y=f(x+a)$, $y=f(ax)$ combined in one expression ($y=f(ax+b)$ as such is not required).





The V meets the line at the marked point; the second intersection (6, 11) lies off-screen. Combine transformations one factor at a time.

Worked example – modulus equation (specification's own example). Solve $|2x-1|=x+5$.

- Case $2x-1\geq 0$ ($x\geq\frac{1}{2}$): $2x-1=x+5 \implies x=6$ ✓ (satisfies $x\geq\frac{1}{2}$).
- Case $2x-1<0$: $-(2x-1)=x+5 \implies -3x=4 \implies x=-\frac{4}{3}$ ✓ ($<\frac{1}{2}$).
- Both solutions are $x=6$ and $x=-\frac{4}{3}$. Check: $|2(6)-1|=11=6+5$ ✓; $|-\frac{8}{3}-1|=\frac{11}{3}=-\frac{4}{3}+5$ ✓.

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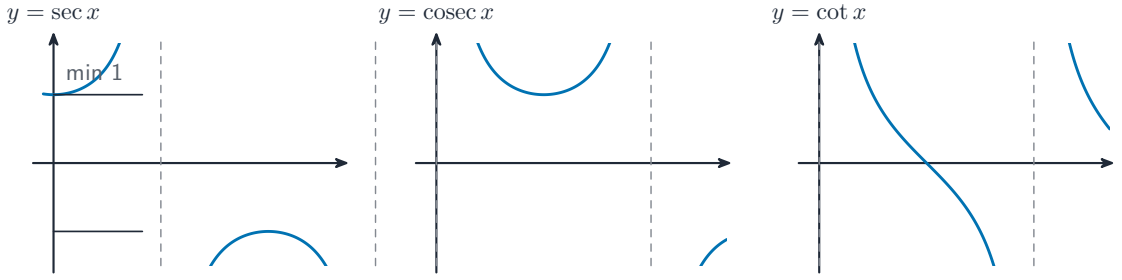
Further trigonometry: sec, cosec, cot and harmonic form

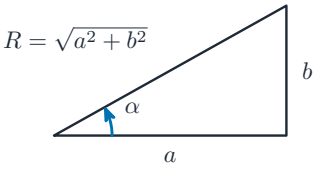
The **reciprocal functions** 倒数三角函数:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Cotangent is defined where $\sin\theta \neq 0$ and equals $1/\tan\theta$ where both expressions are defined. Use the identities $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$. Their graphs have **asymptotes** 渐近线 where the denominator function is zero. The **inverse functions** arcsin, arccos, arctan need **restricted domains** 限制定义域: $\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arccos : [-1,1] \to [0,\pi]$, $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.

Harmonic form 辅助角: $a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$ with $R = \sqrt{a^2+b^2}$ with $R\cos\alpha = a$ and $R\sin\alpha = b$ – use both signs to choose the quadrant. If $R = 0$, the expression is zero. This form is ready to maximise or solve.





$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$ with $\tan\alpha = \frac{b}{a}$: one amplitude R and one phase shift α replace both terms.

Harmonic form packs a cos θ + b sin θ into one cosine of a shifted angle.

Worked example – harmonic form (adapted from Edexcel IAL WMA13 June 2025 Q2). Express $f(x) = 7\cos x - 24\sin x$ in the form $R\cos(x+\alpha)$, $R > 0$, $0 < \alpha < 2\pi$.

- $R = \sqrt{7^2 + 24^2} = \sqrt{625} = 25$.
- Expand $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$ and match: $25\cos\alpha = 7$ and $25\sin\alpha = 24$, so $\tan\alpha = \frac{24}{7}$ and

$$\alpha = 1.287 \text{ (3 d.p.)} \qquad f(x) = 25\cos(x + 1.287)$$

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- Check: $25\cos(0+1.287) = 25(0.28) = 7.0$ ✓ and $-25\sin(1.287) = -24.0$ ✓.

Worked example – identity proof (adapted from Edexcel IAL WMA13 June 2025 Q8a). Prove $\tan 3x \equiv \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

- Write $3x = 2x + x$ and use the **compound angle** 复合角 formula:

$$\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}, \qquad \tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2t}{1 - t^2} \ (t = \tan x)$$

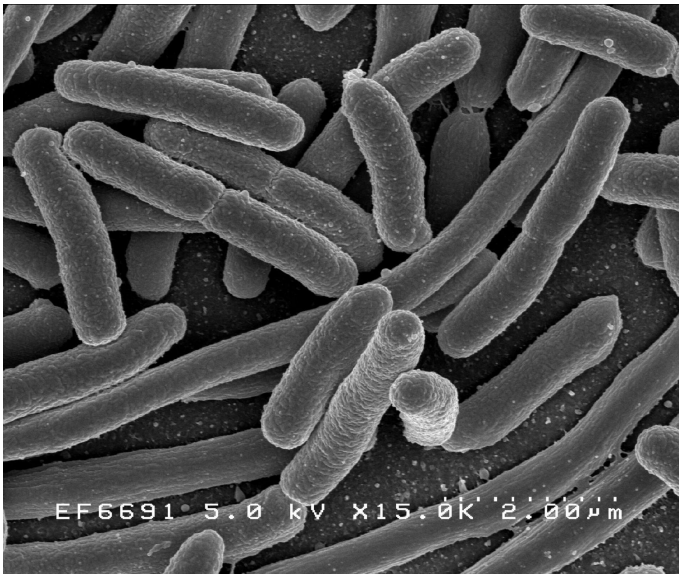
- Substitute:

$$\tan 3x = \frac{\frac{2t}{1-t^2} + t}{1 - \frac{2t^2}{1-t^2}} = \frac{2t + t(1-t^2)}{(1-t^2) - 2t^2} = \frac{3t - t^3}{1 - 3t^2} \qquad \blacksquare$$

- Check at $x = \frac{\pi}{6}$ ($t = \frac{1}{\sqrt{3}}$): numerator $\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}$; denominator $1 - 1 = 0$ – and $\tan\frac{\pi}{2}$ is undefined ✓ consistent.

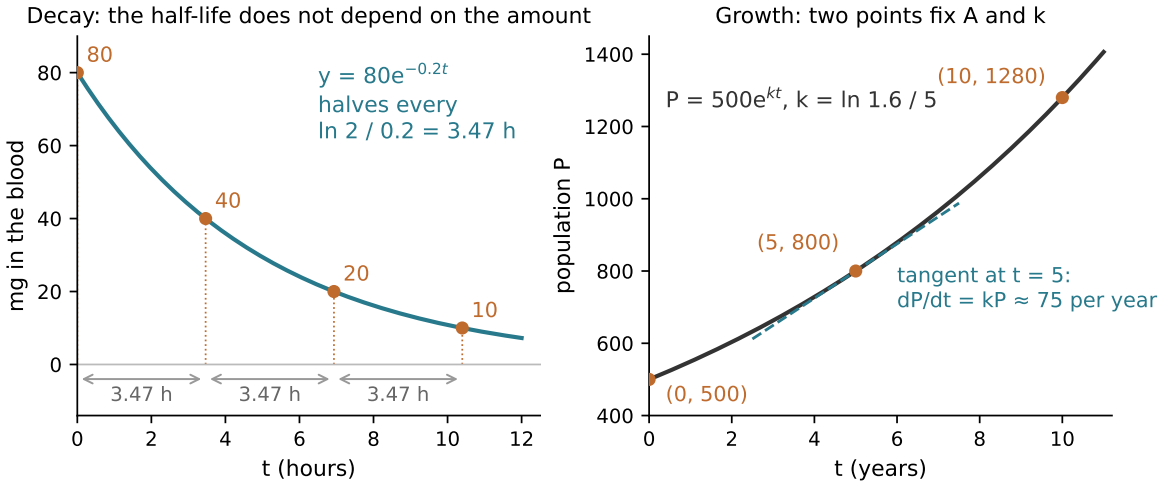
Exponentials, logarithms and log-graph models

$y = e^x$ passes through $(0, 1)$ and grows without bound; $\ln x$ is its inverse with domain $x > 0$ and the y-axis as asymptote. Solve $e^{ax+b} = p$ by taking ln, and $\ln(ax + b) = q$ by exponentiating.



Bacterial growth is roughly exponential while resources last; a log graph turns it into a straight line

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Decay halves in equal times whatever the amount; growth multiplies by 1.6 every 5 years, and the gradient kP is the rate of change

Log-graph models 对数图模型: for $y = ax^n$, taking logs gives $\log y = n \log x + \log a$ – plot $\log y$ against $\log x$ for a straight line with gradient n and intercept $\log a$. For $y = kb^x$, plot $\log y$ against x: gradient $\log b$, intercept $\log k$. Real logarithms require positive arguments and a valid common base; the displayed power-law form uses $a > 0, x > 0$, and the exponential form uses $k > 0, b > 0$.

> **Worked example – recover a power law** (adapted from Edexcel IAL WMA13 June 2025 Q3). A $\log_{10} y$–$\log_{10} x$ line passes through $(-3.5, 0)$ and $(0, -2)$. Express y as px^q.

- Gradient: $\dfrac{0-(-2)}{-3.5-0} = -\dfrac{4}{7}$; intercept -2:

$$\log_{10} y = -\tfrac{4}{7} \log_{10} x - 2$$

- Convert: $y = 10^{-\frac{4}{7}\log_{10} x - 2} = 10^{-2}\, x^{-4/7}$, so

$$y = \frac{1}{100}\, x^{-4/7} \qquad (p = \tfrac{1}{100},\ q = -\tfrac{4}{7})$$

- Check with a point on the line: $\log_{10} x = 0 \Rightarrow x = 1$, $\log_{10} y = -2 \Rightarrow y = \frac{1}{100}$ ✓ and the formula gives $y = \frac{1}{100} \cdot 1 = \frac{1}{100}$ ✓.

> **Worked example – exponential decay model** (adapted from Edexcel IAL WMA13 June 2025 Q6). A processor cools as $\theta = 21 + Ae^{-kt}$ (θ in °C, t in minutes), starting at 75 °C, reaching 25 °C after 5 minutes; find A, k, and the time when the temperature falls at 9 °C per minute.

- At $t = 0$: $75 = 21 + A$, so $A = 54$.
- At $t = 5$: $25 = 21 + 54e^{-5k} \implies e^{-5k} = \frac{4}{54} = \frac{2}{27}$:

$$-5k = \ln \tfrac{2}{27} \implies k = \tfrac{1}{5} \ln \tfrac{27}{2} = 0.521 \text{ (3 s.f.)}$$

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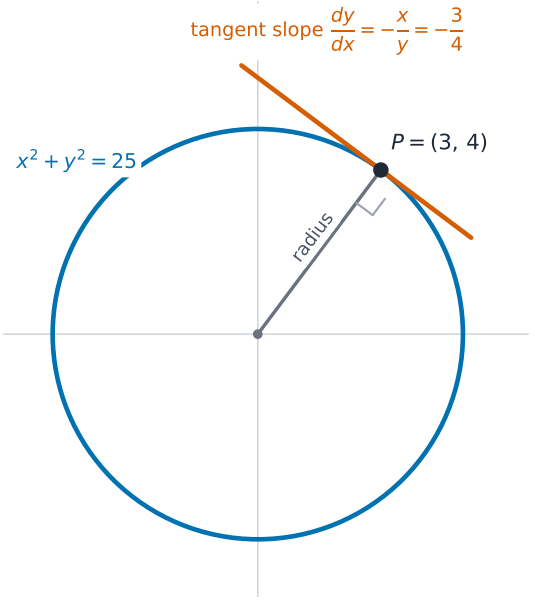
- Rate: $\dfrac{d\theta}{dt} = -54ke^{-kt}$; set equal to -9:

$$e^{-kt} = \frac{9}{54 \times 0.521} = \frac{9}{28.13} \implies T = \frac{\ln(28.13/9)}{0.521} = 2.19 \text{ (2 d.p.)}$$

- The **initial value** 初值 is 75 °C and the **long-run floor** 长期下限 is 21 °C.
- Check: at $T = 2.19$, $\theta = 21 + 54e^{-0.521(2.19)} = 21 + 54(0.320) = 38.3$ °C, falling at $28.13 \times 0.320 = 9.0$ °C/min ✓. The model's floor is 21 °C (room temperature) – sensible for large t.

Differentiation: rules, implicit and models

Differentiate $\sin kx$, $\cos kx$, $\tan kx$ $(\to k\sec^2 kx)$, e^{kx}, $\ln kx$ $(\to \frac{1}{x})$, plus sums and differences.



Implicit differentiation gives the tangent to a circle, perpendicular to the radius

Three rules extend the reach: the **product rule** 乘积法则, the **quotient rule** 商法则 and the **chain rule** 链式法则.

$$\text{product: } (uv)' = u'v + uv' \qquad \text{quotient: } \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \qquad \text{chain: } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\sec x \to \sec x \tan x$, $\operatorname{cosec} x \to -\operatorname{cosec} x \cot x$, $\cot x \to -\operatorname{cosec}^2 x$ follow from the quotient rule. **Implicit differentiation** 隐函数求导 uses $\dfrac{dy}{dx} = 1 \div \dfrac{dx}{dy}$ when x is locally invertible as a differentiable function of y and $dx/dy \neq 0$.

Worked example – product and chain together (from the specification's own examples). Differentiate $e^{-x^2}\sin 3x$ (adapted from Edexcel IAL WMA13 June 2025 Q7).

- Product rule with $u = e^{-x^2}$ (chain: $u' = -2xe^{-x^2}$) and $v = \sin 3x$ ($v' = 3\cos 3x$):

$$\frac{dy}{dx} = -2xe^{-x^2}\sin 3x + 3e^{-x^2}\cos 3x = e^{-x^2}\big(3\cos 3x - 2x\sin 3x\big)$$

- Check at $x = 0$: derivative $= 1 \times (3 - 0) = 3$, and near 0 the curve behaves like $\sin 3x$ with gradient 3 ✓.

Integration: standard forms and recognition

Integrate the standard forms directly, and use **recognition forms** 识别积分 (spotting f' inside the integral):

$$\int \frac{\overbrace{6x}^{f'(x)}}{\underbrace{3x^2+4}_{f(x)}}\,dx = \ln|3x^2+4| + c$$

numerator a multiple of the inner derivative? then check by differentiating

$$\int x\,(3x^2+4)^4\,dx = \frac{(3x^2+4)^5}{30} + c$$

Spot f prime beside f

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c \qquad \int f'(x)\big(f(x)\big)^n\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1} + c \quad (n \neq -1)$$

Trig identities convert $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ into integrable forms ($\cos 2A \equiv 1 - 2\sin^2 A$, $\cos 2A \equiv 2\cos^2 A - 1$; $\tan^2 \equiv \sec^2 - 1$).

Worked example – identity first (adapted from Edexcel IAL WMA13 June 2025 Q5). Find $\int \sin^2 3x\,dx$.

- **Double angle** 二倍角 with $A = 3x$: $\cos 6x \equiv 1 - 2\sin^2 3x$, so $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$:

$$\int \sin^2 3x\,dx = \tfrac{1}{2}\int (1 - \cos 6x)\,dx = \tfrac{1}{2}\left(x - \tfrac{1}{6}\sin 6x\right) + c = \frac{x}{2} - \frac{\sin 6x}{12} + c$$

- Check by differentiating: $\frac{1}{2} - \frac{6\cos 6x}{12} = \frac{1}{2}(1 - \cos 6x) = \sin^2 3x$ ✓.

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Worked example – recognition. Find $\displaystyle\int \frac{6x}{3x^2+4}\,dx$.

- The numerator is the derivative of the denominator ($f' = 6x$, $f = 3x^2+4$):

$$\int \frac{6x}{3x^2+4}\,dx = \ln|3x^2+4| + c$$

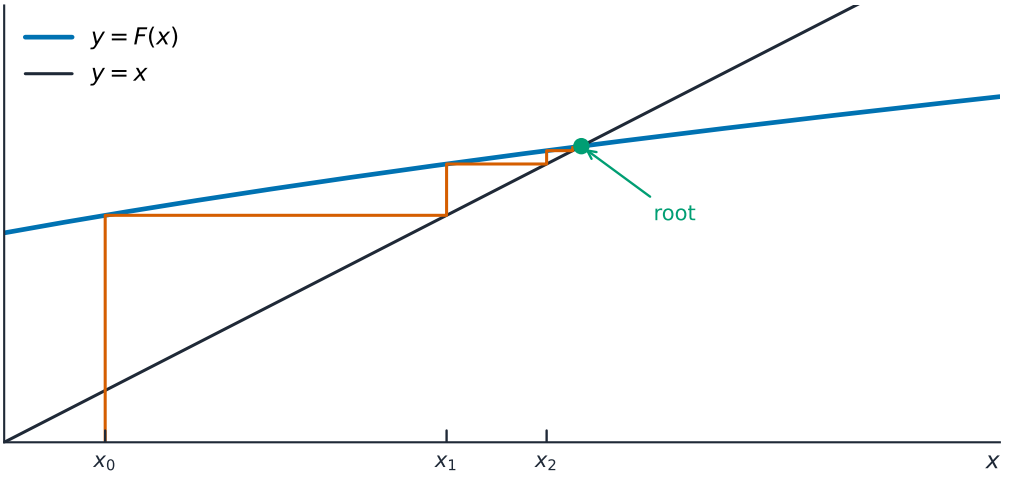
- Check: $\dfrac{d}{dx}\ln(3x^2+4) = \dfrac{6x}{3x^2+4}$ ✓. The powered form works the same way: with $u = 3x^2+4$ and $du = 6x\,dx$,

$$\int x\,(3x^2+4)^4\,dx = \tfrac{1}{6}\int u^4\,du = \frac{(3x^2+4)^5}{30} + c$$

and differentiating $\dfrac{(3x^2+4)^5}{30}$ gives $\dfrac{5(3x^2+4)^4\cdot 6x}{30} = x(3x^2+4)^4$ ✓.

Numerical methods: sign changes and iteration

- **Sign change** 变号: if f is continuous on $[a,b]$ and $f(a)f(b) < 0$, a root lies between a and b. Halve the interval to pin a root to any accuracy.
- **Iteration** 迭代: a rearrangement $x_{n+1} = f(x_n)$ from a start value converges to a **fixed point** 不动点 (or not – always check against a sign-change bound).



$x_{n+1} = F(x_n)$ · converges near the root if $|F'(x)| < 1$ there

Each step goes up to $y = F(x)$ then across to $y = x$; the staircase closes in on the root.

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Worked example – locate and iterate (adapted from Edexcel IAL WMA13 June 2025 Q7). The stationary point of $y = e^{-x^2}\sin 3x$ near $x = 0.4$ satisfies $x = \frac{1}{3}\arctan\frac{3}{2x}$.

(a) With $x_1 = 0.4$:

$$x_2 = \tfrac{1}{3}\arctan\left(\tfrac{3}{0.8}\right) = \tfrac{1}{3}\arctan(3.75) = 0.4367 \qquad x_4 = 0.4307 \text{ (4 d.p.)}$$

- Check the iteration converges: $|x_3 - x_2|$ small and the values settle toward 0.43 ✓.

(b) Show the root is 0.430 to 3 decimal places: define $g(x) = x - \frac{1}{3}\arctan\frac{3}{2x}$ (or the derivative expression from the worked example above):

$$g(0.4295) = -1.1 \times 10^{-3} < 0 \qquad g(0.4305) = +6.4 \times 10^{-5} > 0$$

- g is continuous, and the sign change between 0.4295 and 0.4305 brackets a root, so the root is 0.430 to 3 d.p. ✓.

Checklist before you call this unit done

- Simplify rational expressions; factorise before cancelling; divide by linear or quadratic divisors.
- Work with domain, range, composition and inverses; verify inverses by composing to x.
- Sketch and solve modulus equations by cases with checks; combine two transformations on one graph.
- Use sec, cosec, cot and their identities; know the inverse-function restricted domains.
- Convert $a\cos\theta + b\sin\theta$ to $R\cos(\theta \pm \alpha)$; read off maxima and solve.
- Prove compound and double angle identities; use them forwards and backwards.
- Solve $e^{ax+b} = p$ and $\ln(ax+b) = q$; recover $y = ax^n$ and $y = kb^x$ from log plots.
- Differentiate with product, quotient and chain rules, including sec / cosec / cot; use $\frac{dy}{dx} = 1 \div \frac{dx}{dy}$.
- Model growth and decay; interpret the initial value and long-run floor; critique the model.
- Integrate standard forms; spot $\int f'/f$ and $\int f'(f)^n$; use identities on squared trig functions.
- Locate roots by sign change with a continuity statement; run and check an iteration.

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4 · Pure Mathematics 4

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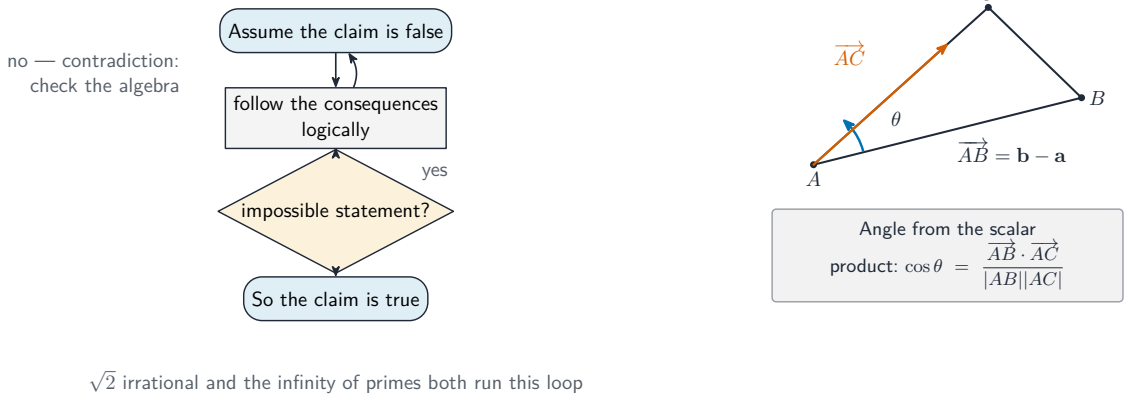
Pure Mathematics 4: proof, parametrics and the full calculus

Pure Mathematics 4 (P4) is the final compulsory pure unit of International A Level Mathematics and International A Level Pure Mathematics. P1–P3 are assumed and may be tested on the WMA14/01 paper (1 hour 30 minutes, 75 marks; same calculator and formula-booklet rules). This reference covers proof by contradiction, partial fractions, parametric equations, the general binomial series, implicit and parametric differentiation, differential equations, the extended integration methods, volumes of revolution, and vectors.

Must-know for P4 (not in the booklet):

Family	Formulae
binomial series	$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \cdots$, valid $\lvert x\rvert < 1$ for any rational n
volume of revolution	$V = \pi \displaystyle\int_a^b y^2\,dx$ about the x-axis (parametric: $\pi \int y^2 \dfrac{dx}{dt}\,dt$)
vectors	column form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; $\lvert\mathbf{a}\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$; unit vector $\dfrac{\mathbf{a}}{\lvert\mathbf{a}\rvert}$; $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Proof by contradiction



Left: proof by contradiction as a flow from assumption to impossibility to conclusion. Right: triangle ABC with position vectors and the angle from the scalar product.

Proof by contradiction 反证法 assumes the opposite of the claim, derives an im-

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possibility, and concludes the original claim. Two classics must be known:

> **Worked example – $\sqrt{2}$ is irrational.** (A number is **irrational** 无理数 when no fraction $\frac{p}{q}$ equals it.) Suppose instead $\sqrt{2} = \frac{p}{q}$ in lowest terms (p, q coprime integers, $q \neq 0$).

- Squaring: $2 = \frac{p^2}{q^2} \implies p^2 = 2q^2$, so p^2 is even, so p is even: write $p = 2r$.
- Then $4r^2 = 2q^2 \implies q^2 = 2r^2$, so q is even too.
- Both p and q even contradicts $\frac{p}{q}$ being in lowest terms. ■

> **Worked example – there are infinitely many primes.** Suppose finitely many: $p_1, \ldots, p_k$.

- Consider $N = p_1 p_2 \cdots p_k + 1$. Dividing N by any p_i leaves remainder 1.
- So N is either prime itself or has a prime factor not in the list – either way a prime outside $\{p_1, \ldots, p_k\}$, contradicting the assumption. ■

> **Worked example – a trigonometric contradiction** (adapted from Edexcel IAL WMA14 June 2025 Q10). Prove that $|\sin x + \cos x| < 1$ for all $90° < x < 180°$.

- Suppose not: some x in the range has $|\sin x + \cos x| \geq 1$. Squaring (both sides non-negative):

$$(\sin x + \cos x)^2 \geq 1 \implies \sin^2 x + 2\sin x \cos x + \cos^2 x \geq 1 \implies 1 + \sin 2x \geq 1 \implies \sin 2x \geq 0$$

- But $90° < x < 180°$ gives $180° < 2x < 360°$, where $\sin 2x < 0$ – a contradiction.
- So no such x exists and $|\sin x + \cos x| < 1$ throughout the range. ■
- Check at $x = 135°$: $|\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}| = 0 < 1$ ✓; at $x = 100°$: $|0.985 - 0.174| = 0.811 < 1$ ✓.

Partial fractions

Decompose 分解 rational functions into **partial fractions** 部分分式. Denominators stay at repeated linear complexity: $(ax+b)(cx+d)(ex+f)$ or $(ax+b)(cx+d)^2$ with the **repeated factor** 重因式 $(cx+d)$ split across powers; quadratic factors are not required. **Improper fractions** 假分式 (numerator degree ≥ denominator's) divide first.

> **Worked example – distinct factors.** Decompose $\dfrac{5x+1}{(x+2)(x-1)}$ (from the P2 sheet family):

$$\frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$$

- Check at $x = 0$: LHS $= \frac{1}{-2} = -\frac{1}{2}$; RHS $= \frac{3}{2} - 2 = -\frac{1}{2}$ ✓.

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$$\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

multiply through: $5x+1 = A(x-1) + B(x+2)$

put $x = 1$: $5(1)+1 = B(1+2)$
$\Rightarrow B = 2$

put $x = -2$: $5(-2)+1 = A(-2-1)$
$\Rightarrow A = 3$

$$\frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$$
check: $3(x-1) + 2(x+2) = 5x+1$

Choose x to make one bracket zero: each substitution gives one constant

Worked example – repeated factor. Decompose $\dfrac{3x^2+7x+6}{x(x+1)^2}$.

- Set up with the repeated factor split across both powers:

$$\frac{3x^2+7x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

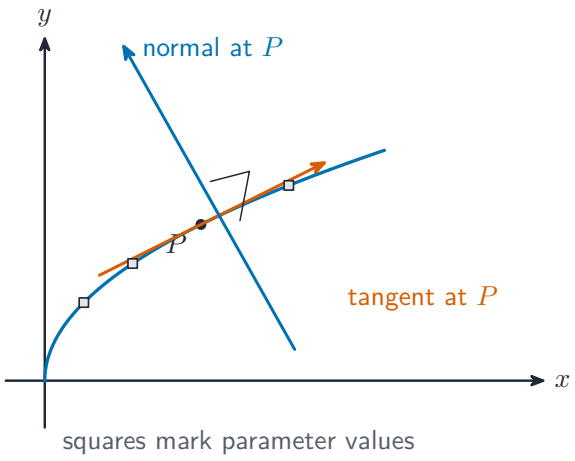
- Multiply through: $3x^2+7x+6 = A(x+1)^2 + Bx(x+1) + Cx$.
- At $x=0$: $6 = A$. At $x=-1$: $3-7+6 = 2 = -C \Rightarrow C = -2$.
- Compare x^2: $3 = A + B \Rightarrow B = -3$.

$$\frac{3x^2+7x+6}{x(x+1)^2} = \frac{6}{x} - \frac{3}{x+1} - \frac{2}{(x+1)^2}$$

- Check at $x=1$: LHS $= \frac{16}{4} = 4$; RHS $= 6 - \frac{3}{2} - \frac{1}{2} = 4$ ✓.

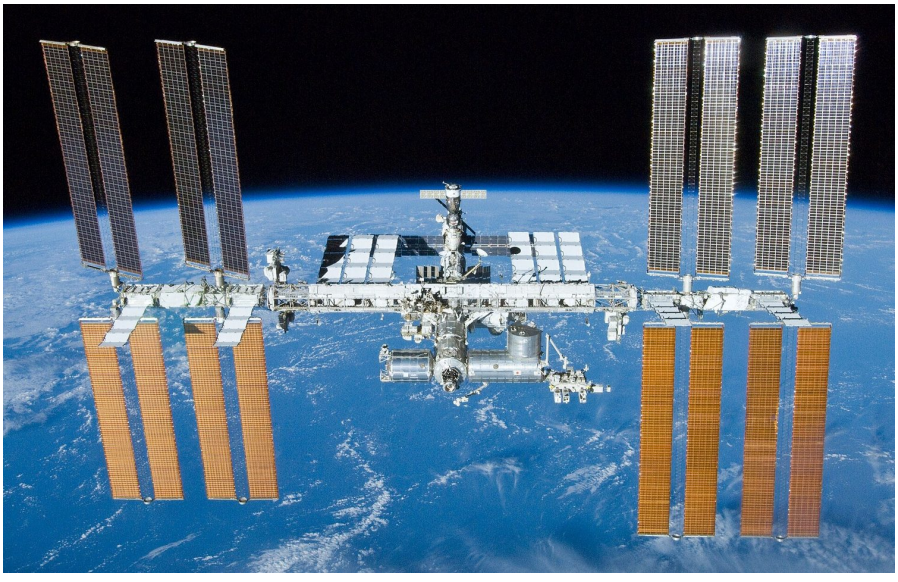
Parametric equations

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Parametric gradient: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$;
the normal takes its negative reciprocal.

A parametric curve with parameter ticks, the tangent at P using $dy/dx = (dy/dt)/(dx/dt)$, and the normal with the negative reciprocal slope.



The ISS in orbit: parametric equations give x and y separately as functions of time

Parametric equations 参数方程 give both coordinates through a parameter t: $x = f(t)$, $y = g(t)$. Convert to **Cartesian** 笛卡尔 form by making t the subject of one equation and substituting into the other (or by exploiting an identity). Never forget the excluded values of x where the parameter map breaks.

Worked example. A curve has $x = 2t$, $y = t^2 - 1$. Find the Cartesian equation.

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- From $x = 2t$: $t = \dfrac{x}{2}$, so

$$y = \left(\frac{x}{2}\right)^2 - 1 = \frac{x^2}{4} - 1$$

- Check at $t = 3$: $(x, y) = (6, 8)$; Cartesian: $36/4 - 1 = 8$ ✓.

> **Worked example – parametric point and normal** (verified facts from Edexcel IAL WMA14 June 2025 Q5). A parametric curve passes through $P(-7, -3)$ at $t = 2$ with $\dfrac{dy}{dx} = -\frac{4}{5}$ there (from $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$).

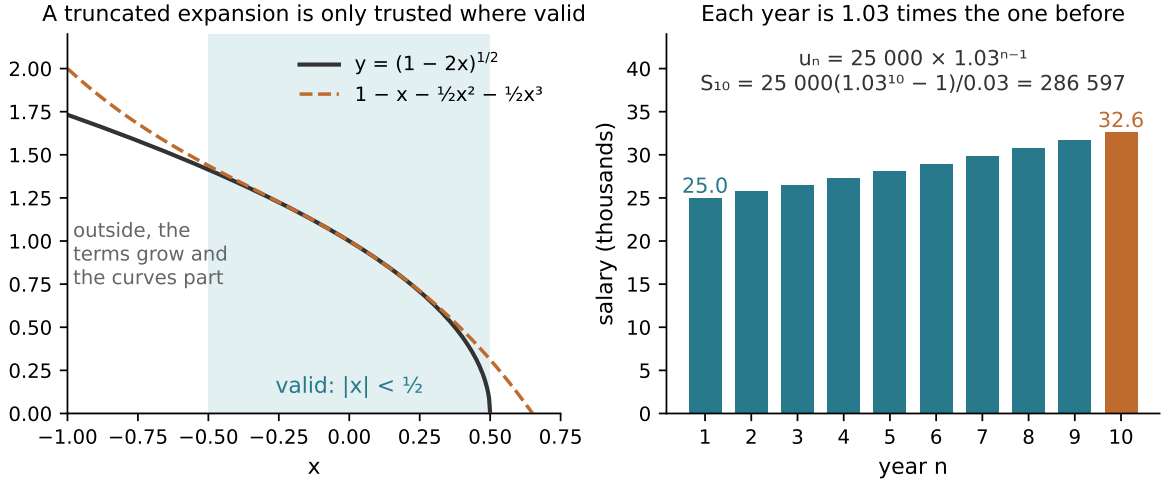
- Normal gradient: $-\dfrac{1}{-4/5} = \dfrac{5}{4}$.

$$y + 3 = \tfrac{5}{4}(x + 7) \implies 5x - 4y + 23 = 0$$

- Check: $5(-7) - 4(-3) + 23 = -35 + 12 + 23 = 0$ ✓.

The binomial series for any rational n

For any **rational** 有理数 n and $|x| < 1$:



The expansion matches the curve only where it is valid, $|x| < ½$; the salary bars grow by the same factor, 1.03, every year

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Factor first: $(a+bx)^n = a^n\left(1 + \frac{b}{a}x\right)^n$, valid for $|x| < \left|\frac{a}{b}\right|$. Decompose rational functions into partial fractions, then expand each piece. Powers do not terminate unless n is a positive integer.

> **Worked example** (adapted from Edexcel IAL WMA14 June 2025 Q8). Expand $\sqrt{4 + x}$ to three non-zero terms.

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- Factor: $\sqrt{4+x} = 2\left(1+\frac{x}{4}\right)^{1/2}$, valid $|x| < 4$, with $n = \frac{1}{2}$:

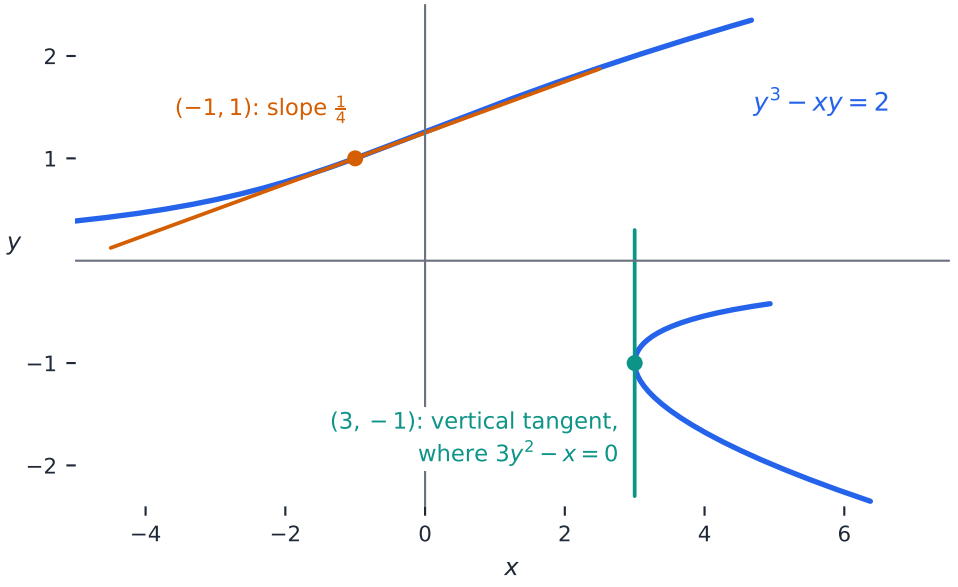
$$\left(1+\frac{x}{4}\right)^{1/2} = 1 + \frac{1}{2}\cdot\frac{x}{4} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{x}{4}\right)^2 + \cdots = 1 + \frac{x}{8} - \frac{x^2}{128} + \cdots$$

$$\sqrt{4+x} = 2 + \frac{x}{4} - \frac{x^2}{64} + \cdots$$

- Check at $x = 1$: series gives $2 + 0.25 - 0.0156 = 2.234$; true $\sqrt{5} = 2.2361\ldots$ ✓ (the third term pulls the two-term value 2.25 down to within 0.002).

Implicit and parametric differentiation

Implicit differentiation 隐函数求导 differentiates both sides with respect to x, treating y as a function of x: $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$, and products such as xy need the product rule: $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$. Then solve for $\frac{dy}{dx}$.



Where the denominator of $\frac{dy}{dx}$ is zero (and the numerator is not), the tangent is vertical

Parametric differentiation: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ when $dx/dt \neq 0$; otherwise check the tangent directly.

> **Worked example – an implicit tangent** (adapted from Edexcel IAL WMA14 June 2025 Q1). The curve $2y^2 - 6xy + 7e^{1-2x} = 27$ passes through the point $\left(\frac{1}{2},\ 4\right)$. Find the tangent there.

- Differentiate term by term:

$$4y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx} - 14\mathrm{e}^{1-2x} = 0 \implies \frac{dy}{dx} = \frac{6y + 14\mathrm{e}^{1-2x}}{4y - 6x}$$

- At $\left(\frac{1}{2}, 4\right)$: $\mathrm{e}^{1-1} = 1$, so

$$\frac{dy}{dx} = \frac{24 + 14}{16 - 3} = \frac{38}{13}$$

- Tangent: $y - 4 = \frac{38}{13}\left(x - \frac{1}{2}\right) \implies 38x - 13y + 33 = 0$.
- Check: $38\left(\frac{1}{2}\right) - 13(4) + 33 = 19 - 52 + 33 = 0$ ✓.

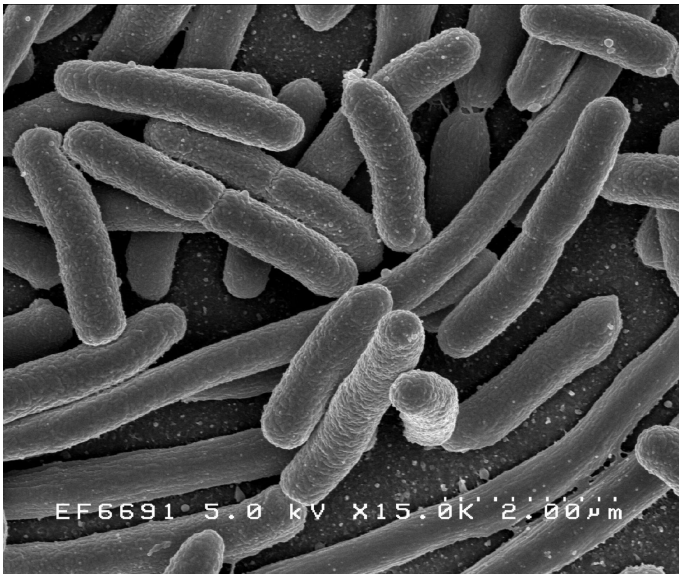
Differential equations and connected rates

Form an equation from a rate description, then solve **separable** 可分离变量 equations:



A cooling cup of coffee: Newton's law of cooling is a first-order differential equation

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E. coli under a microscope: differential equations model exponential growth of populations

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\,dx$$

Check any value divided out while separating variables: constant solutions can be lost. Use a boundary condition for the **particular solution** 特解. **Connected rates of change** 连锁变化率 use the chain rule: $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$.

> **Worked example – a melting sphere** (Edexcel IAL WMA14 June 2025 Q2). A sphere of radius r cm has volume $V = \frac{4}{3}\pi r^3$ and melts with $\dfrac{dV}{dt} = k$ (constant, in cm^3 per second). Show $\dfrac{dr}{dt} \propto \dfrac{1}{r^2}$.

- Differentiate the volume link: $\dfrac{dV}{dr} = 4\pi r^2$.
- Chain rule: $k = \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$, so

$$\frac{dr}{dt} = \frac{k}{4\pi r^2} = \frac{k}{4\pi} \cdot \frac{1}{r^2}$$

- The constant $\dfrac{k}{4\pi}$ multiplies $\dfrac{1}{r^2}$: the rate of decrease of the radius is inversely proportional to r^2. ■
- Interpretation: small spheres vanish disproportionately fast – the last millimetre melts quickest.

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Worked example – a separable equation (adapted from Edexcel IAL WMA14 June 2025 Q3). Solve $y\dfrac{dy}{dx} = 3\sec 2x \tan 2x$, given $y = 4$ at $x = \frac{\pi}{6}$.

- Separate: $\displaystyle\int y\,dy = \int 3\sec 2x \tan 2x\,dx$ (the right side integrates to $\frac{3}{2}\sec 2x$, reversing $\dfrac{d}{dx}\sec 2x = 2\sec 2x \tan 2x$):

$$\frac{y^2}{2} = \frac{3}{2}\sec 2x + c \implies y^2 = 3\sec 2x + C$$

- Substitute $\left(\frac{\pi}{6}, 4\right)$: $16 = 3\sec\frac{\pi}{3} + C = 3(2) + C$, so $C = 10$:

$$y^2 = 3\sec 2x + 10$$

- Check: $3(2) + 10 = 16 = 4^2$ ✓.

Integration: substitution, parts and partial fractions

- **Substitution** 换元积分 (usually given): rewrite the integral in u, adjust dx via $\dfrac{du}{dx}$, integrate, substitute back. It is the chain rule reversed.
- **Integration by parts** 分部积分: $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ – the product rule reversed. Know $\displaystyle\int \ln x\,dx = x\ln x - x + c$ and repeated parts for $\int x^2\mathrm{e}^x dx$-type integrals.
- **Partial fractions** turn a rational integrand into sums of $\dfrac{A}{ax+b}$ (integrating to logs) and $\dfrac{C}{(ax+b)^2}$ (to powers).

Worked example – parts with a repeat in sight. Find $\displaystyle\int \mathrm{e}^x \cos x\,dx$ (needed for volumes of revolution with trig curves).

- Parts with $u = \mathrm{e}^x$, $\frac{dv}{dx} = \cos x$:

$$\int \mathrm{e}^x \cos x\,dx = \mathrm{e}^x \sin x - \int \mathrm{e}^x \sin x\,dx$$

- Parts again on the new integral ($u = \mathrm{e}^x$, $\frac{dv}{dx} = \sin x$):

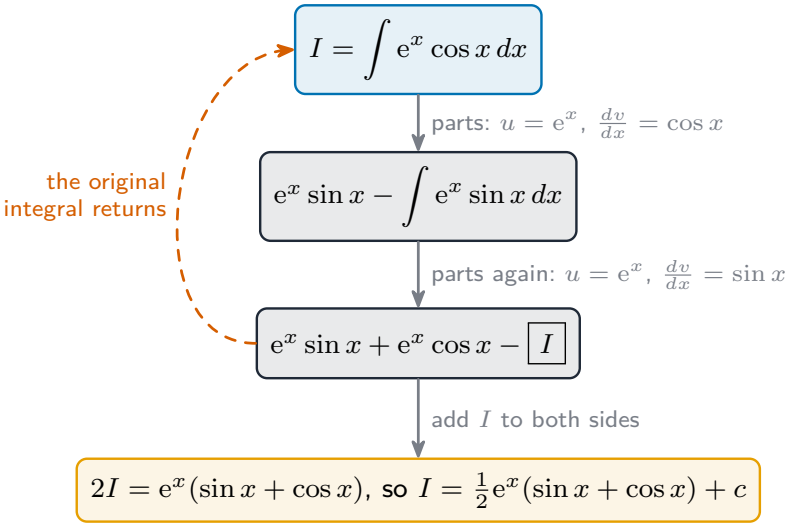
$$= \mathrm{e}^x \sin x + \mathrm{e}^x \cos x - \int \mathrm{e}^x \cos x\,dx$$

- The original integral reappears; collect it:

$$2\int \mathrm{e}^x \cos x\,dx = \mathrm{e}^x(\sin x + \cos x) + c \implies \int \mathrm{e}^x \cos x\,dx = \tfrac{1}{2}\mathrm{e}^x(\sin x + \cos x) + c$$

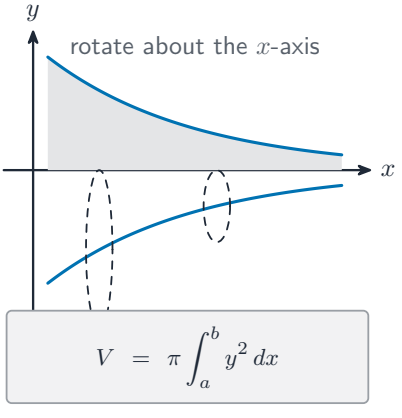
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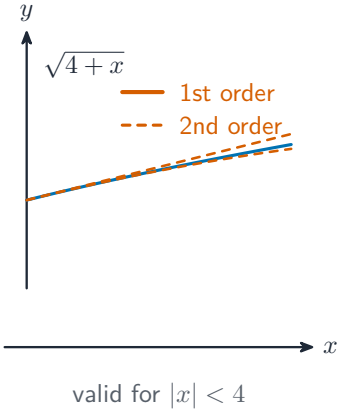
- Check by differentiating: $\frac{1}{2}e^x(\sin x + \cos x) + \frac{1}{2}e^x(\cos x - \sin x) = e^x \cos x$ ✓.



Two rounds of parts bring the original integral back. Collect it: 2I = e to the x (sin x + cos x)

Volumes of revolution and parametric areas





Left: a region rotating about the x-axis with suggested cross-sections and V = π∫y² dx. Right: binomial partial sums of √(4+x) approaching the true curve.

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A potter's vessel is a solid of revolution: rotate a curve about an axis to find its volume

Rotating the region under $y = f(x) \geq 0$ about the x-axis sweeps a solid whose volume is

the **volume of revolution** 旋转体体积

$$V = \pi \int_a^b y^2 \, dx$$

Square the function **before** integrating, use identities to handle $\sin^2/\cos^2$, and split by linearity. With parametric equations, replace $y^2\,dx$ by $y^2 \dfrac{dx}{dt}\,dt$ between parameter values; areas under parametric curves use $y\dfrac{dx}{dt}\,dt$.

> **Worked example – volume with an identity** (structure adapted from Edexcel IAL WMA14 June 2025 Q9). The region under $y = 3\cos x$ for $0 \leq x \leq \frac{\pi}{2}$ rotates about the x-axis; find the exact volume.

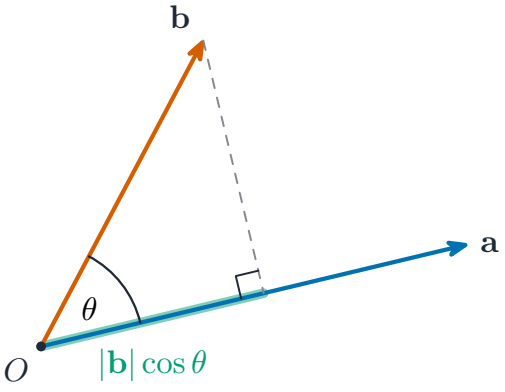
$$V = \pi \int_0^{\pi/2} 9\cos^2 x \, dx = 9\pi \int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2x)\, dx = \frac{9\pi}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{9\pi}{2} \cdot \frac{\pi}{2} = \frac{9\pi^2}{4}$$

- Check: the average of $\cos^2 x$ over a half-turn is $\frac{1}{2}$, so $V \approx 9\pi \times \frac{1}{2} \times \frac{\pi}{2} = \frac{9\pi^2}{4}$ ✓ exactly.

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Vectors

A **vector** 向量 has magnitude and direction, written $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ or as a column.



The scalar product picks out $|\mathbf{b}|\cos\theta$*, how far* $\mathbf{b}$ *reaches along* $\mathbf{a}$*.*

The **magnitude** 模 is $|\mathbf{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$; for $\mathbf{a} \neq \mathbf{0}$, the **unit vector** 单位向量 in its direction is $\mathbf{a}/|\mathbf{a}|$. The zero vector has no direction.

Addition and scalar multiplication have geometric meanings (translations, scaling, **parallel** 平行 when one is a multiple of the other).

The **scalar product** 数量积 $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ gives angles between nonzero vectors through $\cos\theta$. The **position vector** 位置向量 of B relative to A is $\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$.

> **Worked example – a 45° triangle** (Edexcel IAL WMA14 June 2025 Q6, fully verified). $A = \mathbf{i}+2\mathbf{j}-3\mathbf{k}$, $B = 5\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $C = 3\mathbf{i}+\alpha\mathbf{j}+5\mathbf{k}$, and angle $BAC = 45°$.

- $\overrightarrow{AB} = \begin{pmatrix}4\\0\\4\end{pmatrix}$, so $|\overrightarrow{AB}| = \sqrt{32}$; $\overrightarrow{AC} = \begin{pmatrix}2\\\alpha-2\\8\end{pmatrix}$.
- Scalar product: $\overrightarrow{AB}\cdot\overrightarrow{AC} = 8+0+32 = 40$.
- Angle condition with $\cos 45° = \frac{1}{\sqrt{2}}$:

$$\frac{40}{\sqrt{32}\sqrt{68+(\alpha-2)^2}} = \frac{1}{\sqrt{2}} \implies \sqrt{68+(\alpha-2)^2} = \frac{40\sqrt{2}}{\sqrt{32}} = \frac{40\sqrt{2}}{4\sqrt{2}} = 10$$

$$(\alpha-2)^2 = 100-68 = 32 \implies \alpha = 2\pm 4\sqrt{2}$$

- Check: $|\overrightarrow{AB}||\overrightarrow{AC}|\cos 45° = \sqrt{32}\times 10\times\frac{1}{\sqrt{2}} = 4\sqrt{2}\times 10\times\frac{1}{\sqrt{2}} = 40$ ✓ matches the scalar product.

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Checklist before you call this unit done

- Structure contradiction proofs ($\sqrt{2}$, primes) with the assumption stated and the contradiction named.
- Decompose into partial fractions including repeated linear factors; divide improper fractions first; verify by a substituted value.
- Convert parametric $\leftrightarrow$ Cartesian with excluded values; find parametric tangents and normals via $\mathrm{dy/dx} = (\mathrm{dy/dt})/(\mathrm{dx/dt})$.
- Expand $(a + bx)^n$ for rational n with the validity condition; combine expansions of partial fractions.
- Differentiate implicitly (product rule on mixed terms) and parametrically; form differential equations from connected rates.
- Solve separable equations with a boundary condition; check the particular solution.
- Integrate by substitution, parts (including repeats, $\ln x$), and partial fractions.
- Compute volumes of revolution $\pi \int y^2 dx$ and parametric areas via dx/dt.
- Use vectors: magnitude, unit vectors, position vectors; scalar-product angles.

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Mechanics 1

Pearson Edexcel · International A-Level · Mathematics

Mechanics 1: models, motion and forces

Mechanics 1 (M1, paper WME01/01, 1 hour 30 minutes, 75 marks, first assessment June 2019) is the applied unit of International AS Mathematics: P1 and P2 are assumed. Take $g = 9.8\ \mathrm{m\,s^{-2}}$ unless told otherwise, and give inexact answers to 2 or 3 significant figures. This reference covers modelling assumptions, vectors in mechanics, constant-acceleration kinematics, Newton's laws with connected particles, momentum and impulse, friction, statics and moments.

Must-know for M1 (not in the formula booklet – memorise):

Family	Formulae
momentum	momentum $= mv$, impulse $= mv - mu$
suvat	$v = u + at$, $s = ut + \frac{1}{2}at^2$, $s = vt - \frac{1}{2}at^2$, $v^2 = u^2 + 2as$, $s = \frac{1}{2}(u + v)t$
forces	resultant $= ma$ (component-wise), weight $= mg$, friction $F = \mu R$ when moving
vectors	$\lvert a\mathbf{i} + b\mathbf{j}\rvert = \sqrt{a^2 + b^2}$, bearing from atan in the correct quadrant

Mathematical models in mechanics

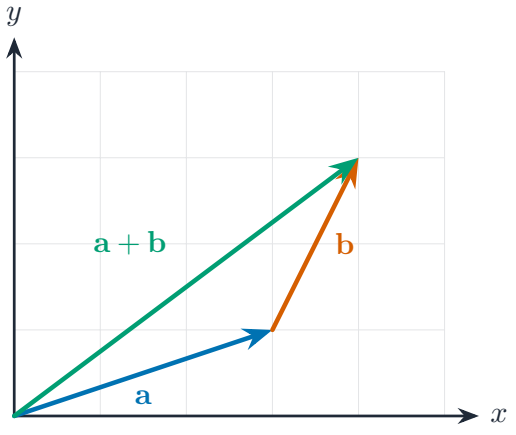
The words in a stem are instructions. A **particle** 粒子 has mass but no size; a **lamina** 薄片 has area but no thickness; a **rod** 杆 has length but no thickness; a **rigid body** 刚体 keeps its shape. **Light** means massless; an unloaded light string over a smooth pulley has the same tension throughout. **Uniform** means constant density, so a uniform rod has its centre of mass at its midpoint. An **inextensible string** 不可伸长绳 keeps its length fixed; a taut string over one fixed pulley links the endpoint speed magnitudes, and a **smooth surface** 光滑表面 versus a **rough surface** 粗糙表面 decide whether friction acts. A **light smooth pulley** 滑轮 changes a tension's direction, not its size. State the assumption you use – "smooth, so no friction" – because the marks are in the modelling as much as the algebra.

Vectors in mechanics

Displacements, velocities, accelerations and forces are vectors: $(3\mathbf{i} - 4\mathbf{j})$ has magnitude 5 and direction $\tan^{-1}(-4/3)$ from the positive x-axis. Resolve into components, work component-wise, and combine at the end. The **resultant** 合力 of $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ is

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$\begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$. With **i** east and **j** north, a **bearing** 方位角 is measured clockwise from north: the vector $-5\mathbf{i} - 12\mathbf{j}$ points south-west, at $\tan^{-1}(5/12) = 22.6°$ west of due south, so its bearing is $180 + 23 = 203°$.



Adding by the triangle law: draw **b** *from the tip of* **a**, *and* **a** + **b** *runs from start to finish.*

Worked example – position vectors and bearings (WME01 June 2025 Q2). Particle A has $\mathbf{r}_A(2) = (-5\mathbf{i} + 16\mathbf{j})$ and $\mathbf{r}_A(5) = (10\mathbf{i} + 4\mathbf{j})$. Velocity $= \dfrac{\Delta \mathbf{r}}{\Delta t} = \dfrac{15\mathbf{i} - 12\mathbf{j}}{3} = 5\mathbf{i} - 4\mathbf{j}$, so $\mathbf{r}_A(t) = (-5\mathbf{i} + 16\mathbf{j}) + (t-2)(5\mathbf{i} - 4\mathbf{j}) = (5t - 15)\mathbf{i} + (24 - 4t)\mathbf{j}$. With B at $\mathbf{s}_B = -5\mathbf{i} + 7\mathbf{j} + t(2\mathbf{i} - 3\mathbf{j})$, at $t = 5$: $\overrightarrow{AB} = (5\mathbf{i} - 8\mathbf{j}) - (10\mathbf{i} + 4\mathbf{j}) = -5\mathbf{i} - 12\mathbf{j}$, bearing 203°.

Kinematics of a particle in a straight line

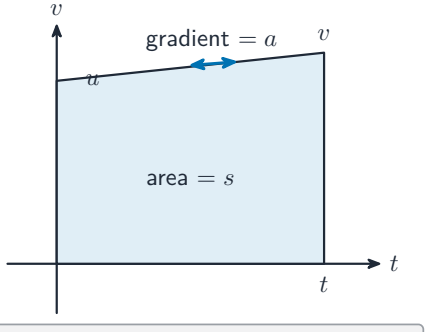
Constant acceleration means the suvat formulae apply once per phase; the discipline is signs. Choose positive, write s, u, v, a, t with signs for each phase, and pick the formula that avoids the unknown you do not want. Graphs: on a velocity–time graph the gradient is acceleration and the area is displacement (area below the axis counts negative); a speed–time graph cannot go negative, so upward motion then downward motion appears as a V.

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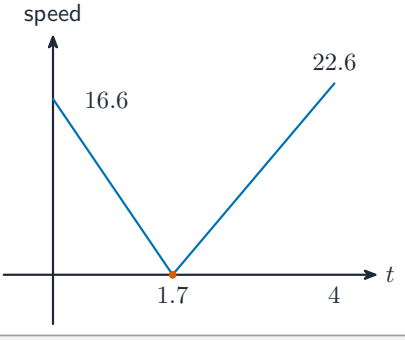


A high-speed train: constant acceleration in a straight line is modelled with the suvat equations

Worked example – up and down (WME01 June 2025 Q3). A ball is projected vertically up from A, 12 m above the ground at B, and takes 4 s to reach B. Taking up positive with A as origin, $s = -12$: $-12 = 4U - 4.9(16)$, so $U = 16.6\ \text{m s}^{-1}$ (2 s.f. 17). Speed at the ground: $v^2 = 16.6^2 + 2(9.8)(12) = 510.8$, so $v = 22.6\ \text{m s}^{-1}$. The speed–time graph is a V: falling from 16.6 to zero at the top (after $16.6/9.8 = 1.7$ s), then rising to 22.6 at $t = 4$ – the downward leg is longer and climbs to a higher speed; both legs have gradient of size g.



velocity–time: gradient accelerates, area displaces



speed–time V: vertex on the axis, downward leg longer

Left: a velocity-time trapezium with gradient a and area s marked. Right: the speed-time V of the worked example, vertex on the time axis, labelled 16.6, 22.6 and t = 4.

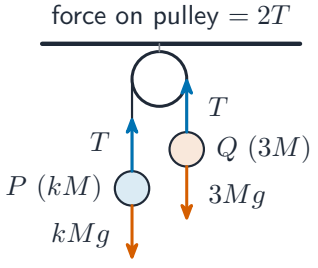
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Newton's laws and connected particles

Newton's second law, applied one way along a line: $F_{\text{net}} = ma$ with every force resolved and signed. For two **connected particles** on a light inextensible string, write an equation of motion for EACH body (same tension, same speed magnitude), then eliminate T. On an inclined plane resolve along and perpendicular to the slope; $\tan\theta = 3/4$ means $\sin\theta = 3/5$, $\cos\theta = 4/5$.

Worked example – two ropes on a barge (WME01 June 2025 Q4). Forces of 800 N at 25° above the line of motion and 500 N at α below it pull a barge straight along the canal. Perpendicular to the motion there is no acceleration: $800\sin 25° = 500\sin\alpha$, so $\sin\alpha = 0.676$ and $\alpha = 43°$. Along the motion, with mass 15000 kg and resistance 750 N: $800\cos 25° + 500\cos\alpha - 750 = 15000a$, giving $a = 0.023\ \text{m s}^{-2}$ – three-decimal answers are expected when the numbers are this small.

Worked example – pulley, start to finish (WME01 June 2025 Q8). Balls P (mass kM) and Q (mass $3M$, $k < 3$) hang over a fixed light smooth pulley and accelerate at $\frac{g}{5}$. Equations of motion: $T - kMg = kM\cdot\frac{g}{5}$ (P up) and $3Mg - T = 3M\cdot\frac{g}{5}$ (Q down). Eliminating T: $3Mg - kMg = \frac{g}{5}(3M + kM)$, so $5(3-k) = 3 + k$ and $k = 2$. The string pulls the pulley down with $2T$: with $M = 0.5$ kg, $T = 1(9.8 + 1.96) = 11.76$ N, so the force is 23.5 N. When Q lands after falling 2.5 m its speed is $\sqrt{2\cdot\frac{g}{5}\cdot 2.5} = \sqrt{g} = 3.13$; if it rebounds at $0.4\ \text{m s}^{-1}$, the impulse on Q is $1.5(3.13 + 0.4) = 5.3$ N s. P then coasts upward against gravity alone for $\frac{\sqrt{g}}{g} = 0.32$ s after a $5/\sqrt{g} = 1.60$ s descent phase: total time to rest ≈ 1.9 s.



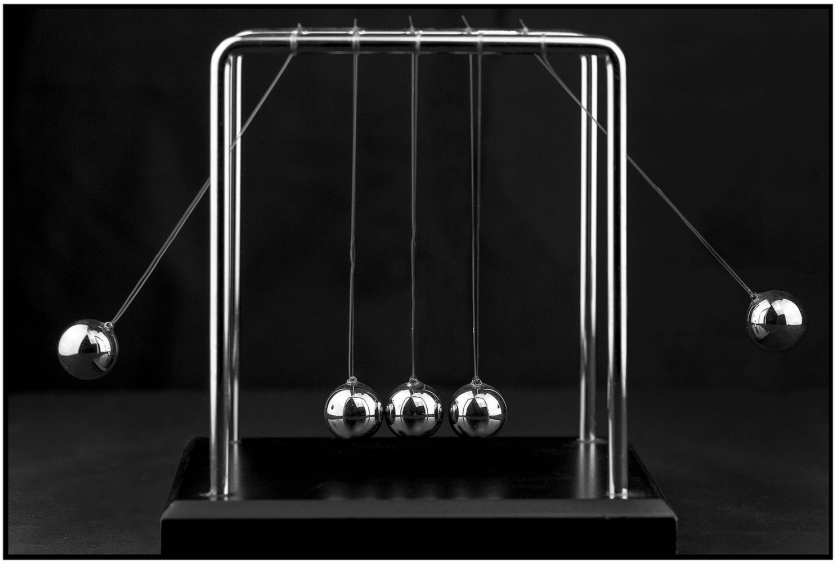
P accelerates up and Q down at $\frac{g}{5}$; same string, same tension: $T - kMg = \frac{kMg}{5}$ and $3Mg - T = \frac{3Mg}{5}$

A light smooth pulley with P of mass kM on the left and Q of mass 3M on the right, tensions T up each side, weights drawn, and the force 2T on the pulley marked.

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Momentum and impulse

Momentum mv and **impulse** 冲量 $I = mv - mu$ are signed along the line. Newton's third law makes the impulse exerted on each of two colliding bodies equal in magnitude and opposite in direction – use one of them, not both. For a direct collision, conservation of momentum $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ needs consistent signs; restitution is NOT in M1.



A Newton's cradle: momentum passes along the line of balls in each collision

> **Worked example – impulse on both bodies (WME01 June 2025 Q1).** P (mass km, speed $5u$) catches Q (mass $3m$, speed $2u$); afterwards P still moves forward at $2u$ and Q at v. The impulse on Q is $4.5mu$: $3m(v - 2u) = 4.5mu$, so $v = 3.5u$. The impulse on P is equal and opposite: $km(5u - 2u) = 4.5mu$, so $k = 1.5$.

Friction

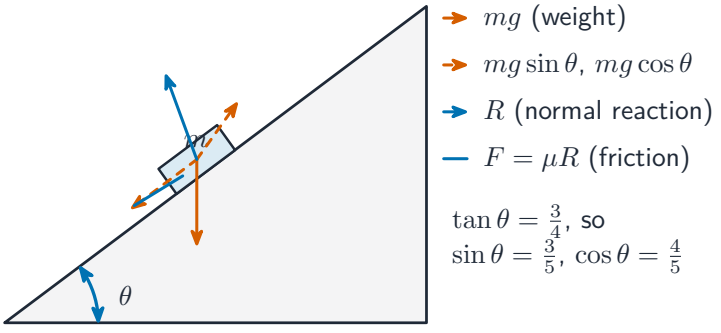
On a **rough** surface the contact force splits into the **normal reaction** 法向反力 R (perpendicular) and friction F (along, opposing motion or its tendency). Moving: $F = \mu R$ with μ the **coefficient of friction** 摩擦系数. In equilibrium $F \leq \mu R$, with equality at limiting equilibrium – the phrase "on the point of moving" licenses $F = \mu R$.

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A climber relies on friction: the limiting value is μR, so steeper rock needs more normal reaction

Worked example – a package on a rough slope (WME01 June 2025 Q6). With $\tan\theta = \frac{3}{4}$ and $\mu = \frac{1}{4}$: perpendicular, $R = mg\cos\theta = \frac{4}{5}mg$; along (up positive, decelerating): $-mg\sin\theta - \frac{1}{4}R = ma$, so $a = -g(\frac{3}{5} + \frac{1}{5}) = -\frac{4}{5}g$ – the deceleration is $\frac{4}{5}g$, and the answer must stay as an exact fraction of g. Held at rest on the slope by a force X up the plane (mass 2 kg): $X + \frac{1}{4}(2g \cdot \frac{4}{5}) = 2g \cdot \frac{3}{5}$, so $X = \frac{4}{5}g = 7.84$ N.



resolve both ways: $R = \frac{4}{5}mg$ and $a = g(\sin\theta + \mu\cos\theta) = \frac{4}{5}g$ when $\mu = \frac{1}{4}$

A block of mass m on a plane inclined at theta with tan theta = 3/4, showing the weight resolved into mg sin theta and mg cos theta, the normal reaction R, and friction F = mu R acting down the slope.

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Statics of a particle

A particle is in **equilibrium** 平衡 when the resultant force is zero: resolve in two perpendicular directions and set both totals to zero, or draw a closed vector triangle. Weight mg down, **normal reaction** perpendicular to the surface, **tension** 张力 along a string away from the body (thrust along a rod pushes), friction along the surface opposing slipping. On rough ground at rest, F is whatever closes the polygon, capped at μR.

> **Worked example – forces in equilibrium (original).** A particle of weight 50 N hangs from two light strings at 40° and 60° to the vertical. Resolving horizontally: $T_1 \sin 40° = T_2 \sin 60°$; vertically: $T_1 \cos 40° + T_2 \cos 60° = 50$. From the first, $T_2 = 0.742 T_1$; then $T_1(0.766 + 0.371) = 50$, so $T_1 = 44$ N and $T_2 = 33$ N (2 s.f.).

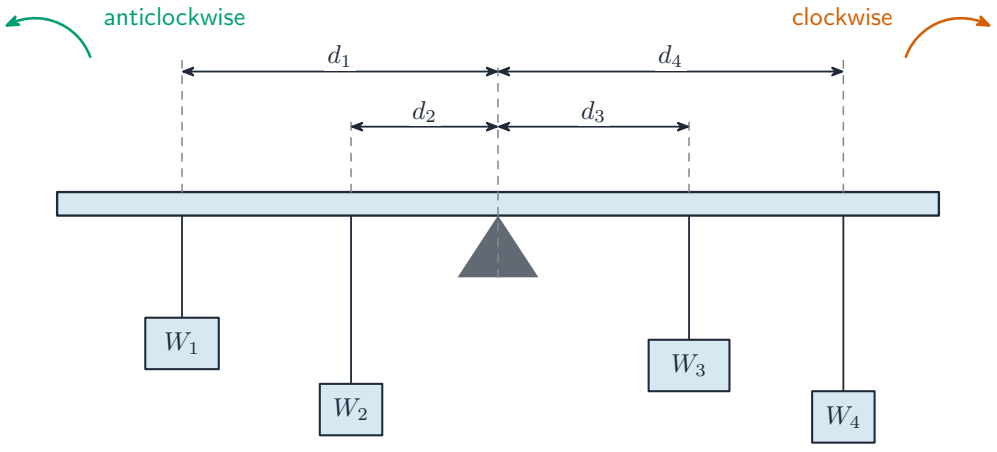
Moments

The **moment of a force** 力矩 about a point is force × perpendicular distance, with a sign per sense. A rigid body in equilibrium has zero resultant moment about ANY point – so pick the pivot that kills the most unknowns. "On the point of tilting about X" means the reaction elsewhere has dropped to zero and moments are taken about X.



A tower crane: the counterweight's moment balances the load's moment about the mast

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Weights on a rule balanced at a pivot – used to test the principle of moments

Worked example – a rod on a ledge (WME01 June 2025 Q5). A non-uniform rod AB of length $3a$ rests on a horizontal ledge with the edge at C, $AC = 2a$, centre of mass d from A. An upward force P at B tilts it about A: moments about A give $3aP = Wd$. A downward force $1.25P$ at B tilts it about C: $1.25Pa = W(2a - d)$. Substituting $P = \frac{Wd}{3a}$ (in units of a): $1.25 \cdot \frac{d}{3} = 2 - d$, so $1.25d = 6a - 3d$, giving $4.25d = 6a$ and $d = \frac{24a}{17}$. The lesson: each tilting case has its own pivot, and only the weight's moment survives unchanged between them.

Checklist before you call this unit done

- Name the modelling assumptions in play (particle, light rod, inextensible string, smooth/rough) and say what each buys you.
- Work with position and velocity vectors component-wise; convert directions to bearings; find relative positions at a common time.
- Solve one-phase and multi-phase suvat problems with signs fixed per phase; read gradients and areas off motion graphs; sketch the speed-time V for up-then-down motion.
- Apply $F = ma$ per body along the line of motion; handle connected particles over pulleys with paired equations; find the force on a pulley as $2T$.
- Use impulse-momentum on ONE body of a collision pair; conserve momentum with consistent signs.
- Split contact forces into R and F; apply $F = \mu R$ when moving or limiting; resolve on inclined planes using $\sin\theta = 3/5$, $\cos\theta = 4/5$ for $\tan\theta = 3/4$.
- Close force polygons for equilibrium in two directions; cap static friction at μR.
- Take moments about the pivot that removes unknowns; switch pivots when the tilting point changes.

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Statistics 1

Pearson Edexcel · International A-Level · Mathematics

Statistics 1: data, probability, regression and distributions

Statistics 1 (S1, paper WST01/01, 1 hour 30 minutes, 75 marks, first assessment June 2019) is the statistics partner of M1 in International AS Mathematics; P1 and P2 are assumed. The formula booklet carries the statistical tables; the formulas below are yours to know. This reference covers data representation and summary, probability, correlation and regression, discrete random variables and the Normal distribution.

Must-know for S1 (not in the booklet):

Family	Formulae
means	$\bar{x} = \dfrac{\sum x}{n}$ or $\dfrac{\sum fx}{\sum f}$
probability	$P(A') = 1 - P(A)$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, $P(A \cap B) = P(A)P(B \mid A)$
coding	$E(aX + b) = aE(X) + b$, $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
Normal	$Z = \dfrac{X - \mu}{\sigma}$ for $X \sim N(\mu, \sigma^2)$

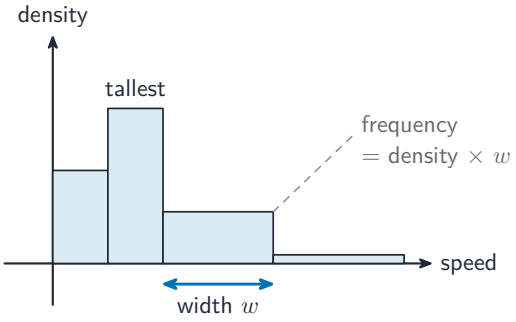
Mathematical models in probability and statistics

A statistical model is a simplified description of a real situation used to represent, analyse and predict. Reasons to use one: it is quicker and cheaper than the real experiment, it can be repeated and refined, and it exposes relationships. Know the standard reasons – "to make predictions" alone is not enough for the mark.

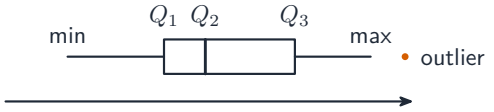
Representing data: histograms, stem and leaf, box plots

In a **histogram** 直方图 the AREA is frequency: bar height is frequency density $\dfrac{\text{frequency}}{\text{class width}}$, so unequal classes get unequal widths, and comparing bars is comparing densities. A **stem and leaf** 茎叶图 keeps the original values (back-to-back compares two groups). A **box plot** 箱线图 shows minimum, lower quartile, median, upper quartile, maximum, with **outliers** 异常值 marked as separate crosses under whatever rule the question states.

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area is frequency: height = f/w for unequal classes



box plot: quartiles and whiskers, outliers as crosses

Left: a histogram of unequal-width classes with frequency density on the vertical axis, one bar annotated frequency = density x width. Right: a box plot with the quartiles labelled and an outlier marked beyond the whisker.

> **Worked example – histogram heights by ratio (WST01 June 2025 Q5a).** Speeds of 225 cars in classes of widths 5, 5, 10, 15, 15 with frequencies 54, 90, 60, 15, 6: the densities are 10.8, 18, 6, 1, 0.4. The tallest bar (25–30, density 18) is drawn 10 cm tall, so the second-tallest (20–25, density 10.8) measures $\frac{10.8}{18} \times 10 = 6$ cm. Beware: the 30–40 bar ALSO has density 6 – only the ratio argument earns the marks, not "60 ÷ 10".

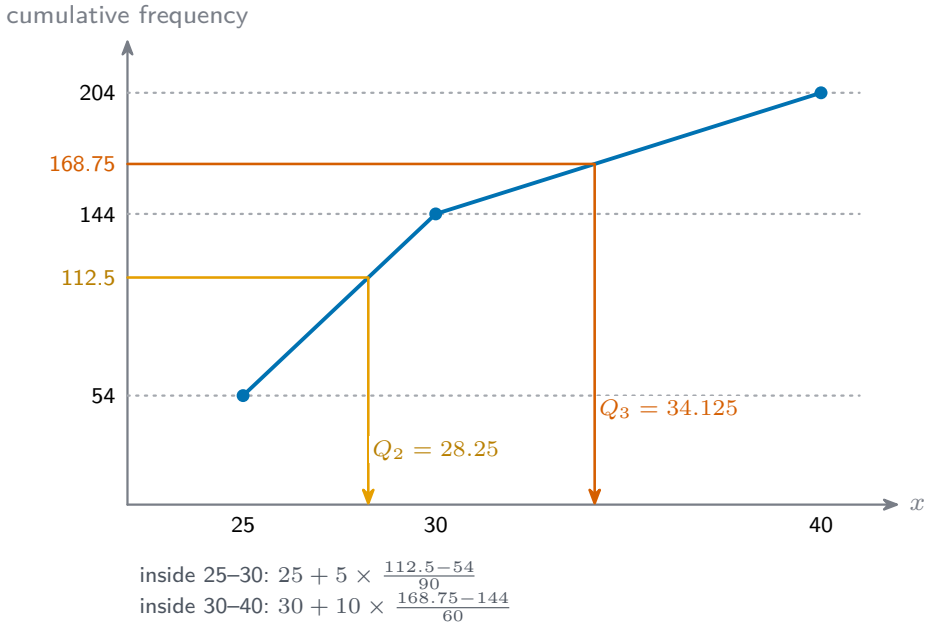
Measures of location and dispersion

For grouped data, estimate the **median** 中位数 or quartiles by linear interpolation inside the containing class: $Q_2 = a + w \times \dfrac{\frac{n}{2} - F}{f}$. The **interquartile range** 四分位距 is $Q_3 - Q_1$; variance is $\sigma^2 = \dfrac{\sum x^2}{n} - \bar{x}^2$ (or $\dfrac{\sum fx^2}{\sum f} - \bar{x}^2$). Combining groups: total $\sum x = \sum n_i \bar{x}_i$, total $\sum x^2$ adds directly, then reapply the formulas. The **standard deviation** 标准差 is the square root of the variance. Coding reverses: if $w = 11.5d$, then $S_{td} = \dfrac{S_{tw}}{11.5}$ and $S_{dd} = \dfrac{S_{ww}}{11.5^2}$.

> **Worked example – combining two classes (WST01 June 2025 Q2).** Class A: $n = 24$, $\bar{x} = 47$, $\sum x^2 = 66876$; class B: $n = 30$, $\bar{x} = b$, $\sum x^2 = 73826$; the 54 combined students average 45. Then $\sum x = 54(45) = 2430$, so $\sum x_B = 2430 - 24(47) = 1302$ and $b = 43.4$. The combined standard deviation: $\sigma = \sqrt{\frac{66876+73826}{54} - 45^2} = \sqrt{580.59\ldots} = 24.1$. If every B mark rises by 2: B's variance is unchanged (shifting moves nothing), the combined mean increases, and the combined SD decreases – B now sits closer to A, so the whole group is less spread.

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Worked example – interpolation quartiles (WST01 June 2025 Q5c–d). With cumulative frequencies 54, 144, 204, 219, 225: the median is the 112.5th value, inside 25–30: $Q_2 = 25 + 5 \times \frac{112.5-54}{90} = 28.25$. Q_3 is the 168.75th, inside 30–40: $Q_3 = 30 + 10 \times \frac{168.75-144}{60} = 34.125$, so with $Q_1 = 25.14$ given, $IQR = 8.985$.



Linear interpolation assumes values spread evenly inside a class: the median 28.25 lies in 25 to 30, the upper quartile 34.125 in 30 to 40

Skewness

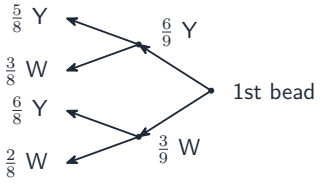
Compare distances: $(Q_3 - Q_2)$ versus $(Q_2 - Q_1)$, or mean versus median. Above, $5.875 > 3.11$ (and mean 30.6 > median 28.25), so the speeds are **positively skewed** 右偏 (right-skewed) – which is also why a Normal model would be refused.

Probability

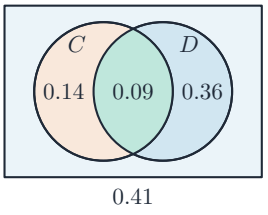
Work from the **sample space** 样本空间: listing outcomes beats formula-grabbing. For any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $P(A \cap B) = P(A)P(B \mid A)$; **independence** 独立 means $P(B \mid A) = P(B)$, giving the product rule. **Conditional probability** 条件概率 reverses the tree: $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$ when $P(A) > 0$. Without replacement, reduce the total remaining count by one; reduce a category count only when the item drawn came from that category.



Dice: equally likely outcomes, the starting point for probability



without replacement: second-layer fractions drop by one top and bottom



$P(C \cup D) = P(C) + P(D) - P(C \cap D)$ fills the regions

Left: a tree diagram for two beads drawn without replacement, first-layer fractions 6/9 and 3/9, second layer 5/8, 3/8 and 6/8, 2/8. Right: a Venn diagram of events C and D with P(C) = 0.23, P(D) = 0.45, intersection 0.09.

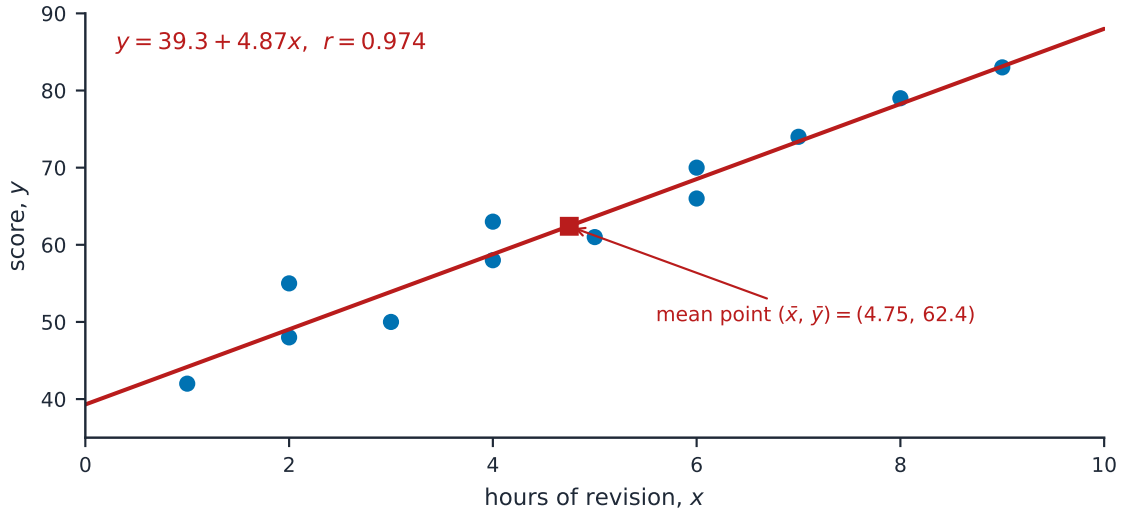
Worked example – Venn to conditional (WST01 June 2025 Q4). $P(C \cup D) = 0.59$, $P(D) = 0.45$, $P(C \mid D) = 0.2$. Then $P(C \cap D) = 0.2 \times 0.45 = 0.09$ and $P(C) = 0.59 + 0.09 - 0.45 = 0.23$ – fill the Venn diagram and both answers fall out.

Worked example – a tree with reversal (WST01 June 2025 Q7). Bag: 3 white, 6 yellow; two drawn without replacement. Tree: $\frac{6}{9}, \frac{3}{9}$, then $\frac{5}{8}, \frac{3}{8}$ after yellow and $\frac{6}{8}, \frac{2}{8}$ after white. Exactly one yellow: $\frac{6}{9} \cdot \frac{3}{8} + \frac{3}{9} \cdot \frac{6}{8} = \frac{1}{2}$. At least one white: $1 - \frac{6}{9} \cdot \frac{5}{8} = \frac{7}{12}$. Given at least one white, both white: $\dfrac{\frac{3}{9} \cdot \frac{2}{8}}{\frac{7}{12}} = \dfrac{1}{12} \cdot \dfrac{12}{7} = \dfrac{1}{7}$.

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Correlation and regression

The **product moment correlation coefficient** 相关系数 $r = \dfrac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$ measures LINEAR association ($-1 \le r \le 1$); interpret in context. The least-squares **regression line** 回归线 of y on x is $y = a+bx$ with $b = \dfrac{S_{xy}}{S_{xx}}$ and $a = \bar{y} - b\bar{x}$ – only usable to predict y FROM x, and only within the data range (extrapolation is the flagged danger). Swap x and y and the line is different.



Regression line of y on x through the mean point

> **Worked example – coded regression (WST01 June 2025 Q3).** With $w = 11.5d$, $S_{tt} = 26.43$, $S_{tw} = -91.55$, $\sum w = 339.25$, $\sum t = 9.1$, $\sum w^2 = 10036.45$ over 12 days: $S_{ww} = 10036.45 - \frac{339.25^2}{12} = 445.57$; the pmcc $r = \dfrac{-91.55}{\sqrt{26.43 \times 445.57}} = -0.844$ – warmer days, less gas. The regression of w on t: $b = \frac{-91.55}{26.43} = -3.46$, $a = \frac{339.25}{12} - (-3.46)\frac{9.1}{12} = 30.9$, so $w = -3.46t + 30.9$; divide through by 11.5 for d on t: $d = -0.301t + 2.69$. At $t = 2$: $d = 2.09$ m^3, and each extra degree means about 0.3 m^3 less gas.

Discrete random variables

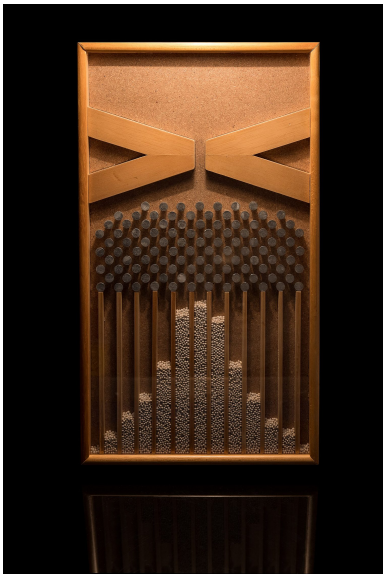
A **discrete random variable** 离散随机变量 takes isolated values with probabilities $p(x) = P(X = x)$ summing to 1; its **probability function** 概率函数 table is the object. **Expectation** 期望 $E(X) = \sum xp(x)$, $E(X^2) = \sum x^2p(x)$, **variance** 方差 $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$. The cumulative function $F(x_0) = P(X \le x_0)$ adds probabilities up to x_0. The discrete UNIFORM distribution on $1, \ldots, n$ has $E(X) = \frac{n+1}{2}$ and $\mathrm{Var}(X) = \frac{n^2-1}{12}$.

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Worked example – the full routine (WST01 June 2025 Q1). X with probabilities 0.3, 0.2, 0.35, 0.15 on 1–4. $F(3.5) = 0.3 + 0.2 + 0.35 = 0.85$. The inequality $3X - 3 < X + 2$ solves to $X < 2.5$, so $P = 0.5$ – solve for X FIRST, then add. $E(X) = 2.35$, $E(X^2) = 6.65$, $\text{Var}(X) = 6.65 - 2.35^2 = 1.1275$; and $\text{Var}(5 - 2X) = 4(1.1275) = 4.51$ by the coding rule.

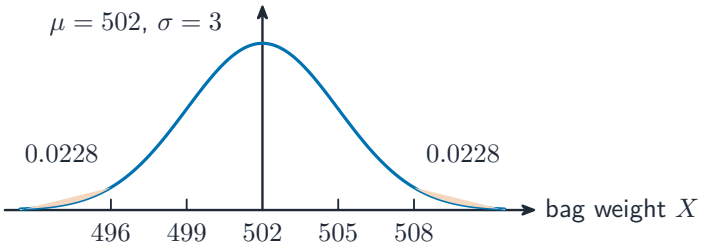
The Normal distribution

For the **Normal distribution** 正态分布 $X \sim N(\mu, \sigma^2)$, standardise: $Z = \dfrac{X - \mu}{\sigma}$ then read the table. Symmetry does half the work: $P(X > \mu + 2\sigma) = P(Z > 2) = 0.0228$ and $P(\mu - 2\sigma < X < \mu + 2\sigma) = 1 - 2(0.0228) = 0.9544$. Reverse problems give z from the table, then $x = \mu + z\sigma$; two conditions on the same k give simultaneous equations.



A Galton board: balls falling through pins pile up into the bell-shaped normal curve

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standardise $Z = \dfrac{X - \mu}{\sigma}$: tails at $z = \pm 2$ hold 0.0228 each

The Normal curve centred at mu = 502 with standard-deviation ticks at 496, 499, 505, 508; the tails beyond 496 and 508 shaded (about 2.28 percent each), and the standardising arrow Z = (X minus mu)/sigma.

Worked example – bags of flour (WST01 June 2025 Q6). Small bags $X \sim N(502, 3^2)$: $P(X > 508) = P(Z > 2) = 0.0228$. Of 4 bags, exactly 2 over 508: $6(0.0228)^2(0.9772)^2 = 0.00298$. Large bags $Y \sim N(1024, \sigma^2)$ with 85th percentile 1038.51: $z_{0.85} = 1.0364$, so $\sigma = \frac{14.51}{1.0364} = 14$. If $P(X > k) = P(Y < 2k)$, then $\frac{k-502}{3} = \frac{1024-2k}{14}$, so $14k - 7028 = 3072 - 6k$, $k = 505$, and $p = P(Z > 1) = 0.159$.

Checklist before you call this unit done

- Give several reasons why statistical models are used; keep "cheap and quick" plus one more.
- Compute frequency densities for unequal classes; compare histogram bars by ratio; interpolate quartiles inside the containing class; mark outliers under the stated rule.
- Combine data sets through $\sum x$ and $\sum x^2$; reverse coding on S statistics; predict the effect of shifting or scaling on mean, variance and SD.
- Solve probability questions from Venn diagrams and trees, with and without replacement; reverse conditionals with $P(A \cap B)/P(A)$.
- Compute r from S statistics and interpret it; fit $y = a + bx$; predict only within range; pass coded data back to the original variable.
- Run the discrete DRV routine: probabilities sum to 1, solve inequalities for X first, $E(X)$, $E(X^2)$, variance, linear coding; quote the uniform results.
- Standardise both directions for Normal questions; use symmetry; finish two-condition problems with simultaneous equations.

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Mechanics 2

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Mechanics 2: projectiles, centres of mass, energy and collisions

Mechanics 2 (M2, paper WME02/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) is the IAL applied unit that builds on M1: P1–P4 and M1 are assumed. Take $g = 9.8\ \mathrm{m\,s^{-2}}$ unless the question's model says otherwise (one June 2025 question uses 10). This reference covers variable-acceleration kinematics, centres of mass, work, energy and power, collisions with restitution, and statics of rigid bodies.

Must-know for M2 (not in the booklet):

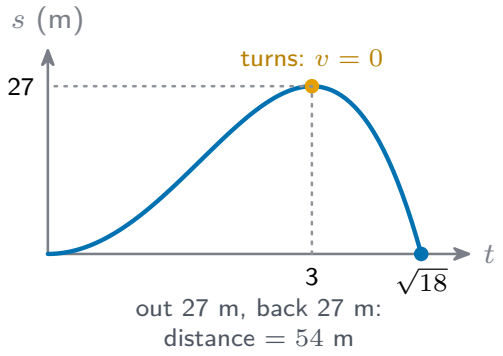
Family	Formulae
energy	**kinetic energy** 动能 $= \frac{1}{2}mv^2$, **potential energy** 势能 $= mgh$, work $= Fs$, power $= Fv = \frac{dW}{dt}$
calculus kinematics	$v = \int a\,dt$, $s = \int v\,dt$ (constants closed by initial conditions)
restitution	$e = \dfrac{v_B - v_A}{u_A - u_B}$ along the line, $0 \le e \le 1$
M1 carryover	momentum mv, impulse $= mv - mu$, $F = \mu R$, moments

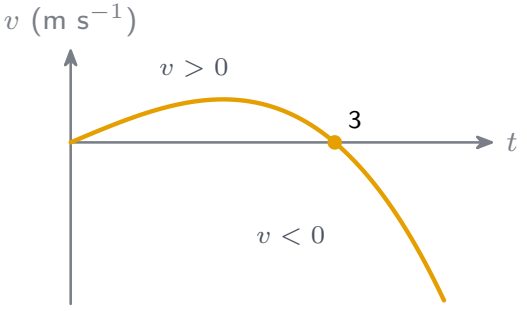
Kinematics with variable acceleration

When acceleration is a function of time, **integrate**: 积分 $v = \int a\,dt$ and $s = \int v\,dt$, with the constants fixed by the initial conditions ("at rest at O" kills both). For total DISTANCE when the particle turns around, find the stopping time ($v = 0$) and add both legs. Position vectors differentiate component-wise: $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, $\mathbf{a} = \frac{d\mathbf{v}}{dt}$.

> **Worked example – integration with a trap (WME02 June 2025 Q1).** With $a = k(3-t^2)$ from rest at O and $v(1.5) = 13.5$: $v = k(3t - \frac{t^3}{3})$ (constant zero by rest), and $k(4.5 - 1.125) = 3.375k = 13.5$ gives $k = 4$. Then $s = 6t^2 - \frac{t^4}{3} = \frac{t^2}{3}(18 - t^2)$. For the total distance back to O: $v = \frac{4t}{3}(9 - t^2) = 0$ at $t = 3$, $s(3) = 27$, so the particle travels $27 + 27 = 54$ m – the return leg is a second 27, not a subtraction.

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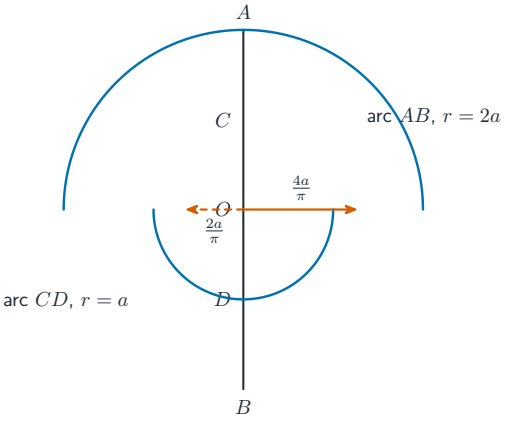




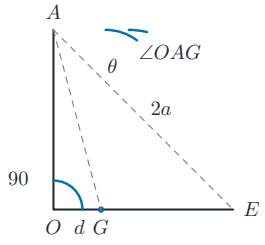
The particle stops at $t = 3$, 27 m from O, then comes back: the total distance is $27 + 27 = 54$ m, while the displacement at return is 0

Worked example – vector velocity then impulse (WME02 June 2025 Q2). With $\mathbf{r} = 4\sqrt{t}\,\mathbf{i} - 3t\mathbf{j}$ (metres, t seconds), mass 3 kg: $\mathbf{v} = \frac{d\mathbf{r}}{dt} = 2t^{-1/2}\mathbf{i} - 3\mathbf{j}$, so at $t = 4$ the velocity is $\mathbf{i} - 3\mathbf{j}$. An impulse $(6\mathbf{i} - 3\mathbf{j})$ N s gives $3(\mathbf{v}' - (\mathbf{i} - 3\mathbf{j})) = (6\mathbf{i} - 3\mathbf{j})$, so $\mathbf{v}' = 3\mathbf{i} - 4\mathbf{j}$ and the speed is 5 m s^{-1} – differentiate FIRST, apply the impulse at the stated instant second.

Centres of mass: particles, laminas and wire frameworks



signed moments: $2\pi a \cdot \frac{4a}{\pi} - \pi a \cdot \frac{2a}{\pi} = 6a^2$ over $(3\pi + 2)a$



$\tan\angle OAG = \frac{3}{3\pi+2}$ and $\theta = 45 - \angle OAG$

Left: the S-shaped wire framework - straight line A-C-O-D-B with the outer semicircular arc of radius 2a to the right and the inner arc of radius a to the left, COM distances 4a/pi and 2a/pi marked from the line. Right: the construction triangle A-O-E with the right angle at O, G on OE at distance d, the dashed hanging line AG, and the angles theta and OAG at A.

For particles: the **centre of mass** 质心 is $\bar{x} = \dfrac{\sum mx}{\sum m}$. For uniform bodies use the table method – split into pieces whose COM you know (booklet: semicircular wire $\frac{2r}{\pi}$, semicircular lamina $\frac{4r}{3\pi}$, sector, arc), weight each piece by MASS (lamina: area; wire: length), and take moments about a convenient line. Pieces on opposite sides of the line carry opposite signs. Composite laminas subtract holes as negative areas.

> **Worked example – an S-shaped wire framework 线框架 (WME02 June 2025 Q3).** Two semicircular wire arcs, AB of radius $2a$ and CD of radius a, both centred at O on opposite sides of the straight wire ACDB ($AC = DB = a$). Table (about the line AB):

piece	length	COM from AB	moment
arc AB	$2\pi a$	$\frac{2(2a)}{\pi} = \frac{4a}{\pi}$	$8a^2$
arc CD	πa	$-\frac{2a}{\pi}$ (opposite side)	$-2a^2$
straights	$2a$	0	0

Total length $(3\pi + 2)a$; moments $6a^2$; so the centre of mass is $\dfrac{6a}{3\pi + 2}$ from AB. Suspended from A with E on the arc where $\angle AOE = 90°$: the symmetry line OE carries G, so $\tan\angle OAG = \dfrac{3}{3\pi + 2}$, and since $OA = OE$ makes $\angle OAE = 45°$, the angle θ of AE below the vertical satisfies $\theta = 45° - \angle OAG$, giving $\tan\theta = \dfrac{3\pi - 1}{3\pi + 5}$.

Suspending laminas and frameworks

A freely suspended body hangs with its centre of mass vertically below the point of suspension. The angle asked for is usually between a drawn chord of the body and the vertical: find the angle at the suspension point by right-angled trigonometry with the COM distance, then adjust by any geometry angle (isosceles triangles are the standard source of 45°, 60°).

Work, energy and power

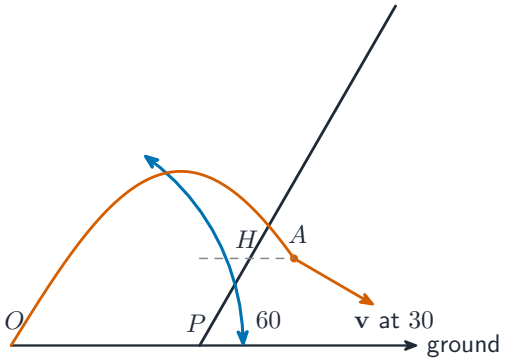
Work done 功 by a constant force: Fs along the motion (negative against it); gravity does $\pm mgh$. The **work-energy principle** 功能原理: total work = change in KE. **conservation of mechanical energy** 机械能守恒: energy is conserved when only gravity does work; resistance enters as work $-Rs$. **power** 功率 $P = Fv$: at constant speed the driving force balances resistance plus gravity-components, so $v = \frac{P}{F}$; at a given speed the excess force accelerates.

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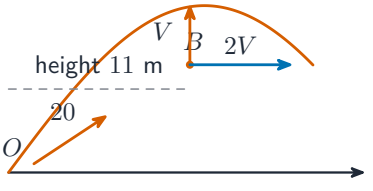


A roller coaster: potential energy at the top becomes kinetic energy lower down, less the work done against resistance

Worked example – a car on a slope (WME02 June 2025 Q4). A 750 kg car climbs a slope with $\sin\alpha = \frac{1}{7}$, resistance 200 N, engine at 20 kW. Constant speed: $\frac{20000}{V} = 750(9.8)\frac{1}{7} + 200 = 1250$, so $V = 16\ \mathrm{m\,s^{-1}}$. At $10\ \mathrm{m\,s^{-1}}$: $2000 - 1250 = 750a$, so $a = 1\ \mathrm{m\,s^{-2}}$. Engine off at A at $10\ \mathrm{m\,s^{-1}}$: work-energy gives $\frac{1}{2}(750)(100) = 750g(d\sin\alpha) + 200d$, i.e. $37500 = 1250d$ and $d = 30$ m – the resistance acts over the SLOPE distance d, gravity over the height $d\sin\alpha$.



perpendicular impact fixes the velocity direction at A



energy: $\frac{1}{2}m(20^2) = \frac{1}{2}m(5V^2) + mg(11)$

Left: a projectile trajectory meeting a 60-degree plane perpendicularly at A, with the velocity arrow at 30 degrees below the horizontal and the height H marked. Right: the same trajectory with the velocity components V (vertical) and 2V (horizontal) marked at the 11 m point.

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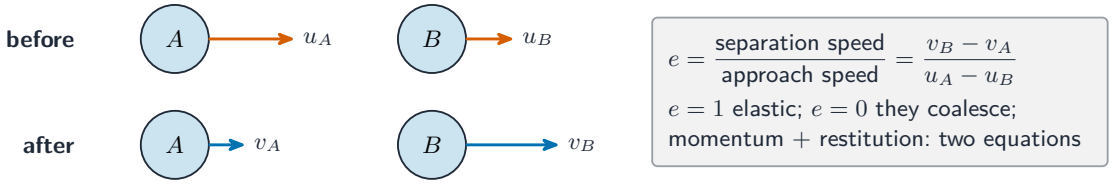
Worked example – a projectile closed by energy (WME02 June 2025 Q7). A **projectile** 抛体 on a parabolic **trajectory** 轨迹 leaves O at 20 $\mathrm{m\,s^{-1}}$ ($g = 10$) is 11 m up when its vertical component is V and its horizontal component $2V$. Energy: $200 = \frac{1}{2}(V^2 + 4V^2) + 110$, so $V = 6$ (components 6 and 12). It meets a plane at 60° to the horizontal PERPENDICULARLY, so the velocity is at 30° below horizontal: speed $= \frac{12}{\cos 30°} = 8\sqrt{3}$, and energy from O gives $200 = \frac{1}{2}(192) + 10H$, so $H = 10.4 \to 10$ m (2 s.f.).

Collisions and restitution

Newton's experimental law, the **coefficient of restitution** 恢复系数: $e = \dfrac{v_B - v_A}{u_A - u_B}$ (separation speed over approach speed, $0 \le e \le 1$; $e = 1$ elastic, $e = 0$ they stick). With conservation of momentum, two equations solve any direct collision. The KE loss always factorises with $(1 - e^2)$: loss $= \frac{m_A m_B}{2(m_A + m_B)}(1 - e^2)(u_A - u_B)^2$ – and the impulse on either body comes from its own momentum change. A rebound from a fixed wall multiplies the speed by e outright.



A bouncing ball: each rebound speed is e times the impact speed, so the bounces shrink



Newton's experimental law: separation speed = e × approach speed

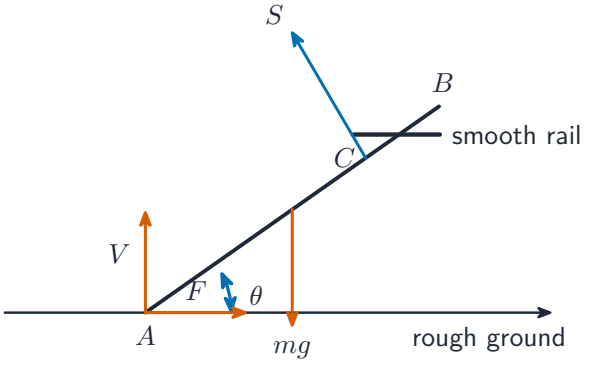
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Worked example – the full collision story (WME02 June 2025 Q5). P (mass $2m$, speed u) strikes Q (mass m) at rest, restitution e. Momentum: $2u = 2v_P + v_Q$; restitution: $eu = v_Q - v_P$. Solving: $v_Q = \frac{2(1+e)u}{3}$, $v_P = \frac{(2-e)u}{3}$. The KE after is $\frac{mu^2}{3}(2+e^2)$, so the loss is $\frac{mu^2}{3}(1-e^2)$ – the factor $k = \frac{1}{3}$. If the impulse on Q is $\frac{8}{9}mu$: $m \cdot \frac{2(1+e)u}{3} = \frac{8mu}{9}$ gives $e = \frac{1}{3}$. Q then rebounds from a wall with restitution f to match P's speed $\frac{5}{9}u$: $f \cdot \frac{8}{9}u = \frac{5}{9}u$, so $f = \frac{5}{8}$.

Successive impacts

Three-body and wall chains are bookkeeping: solve the first collision completely, carry the surviving speeds into the next impact, and only equate quantities the question ties together ("same speed", "at rest"). Wall rebounds reverse direction and scale speed by e; keep a signed diagram per stage.

Statics of rigid bodies



moments about A: $S(1.5a) = mga\cos\theta$,
and $F = S\sin\theta$ at limiting equilibrium

A uniform rod AB leaning at angle theta with end A on rough ground, resting against a smooth horizontal rail at C (AC = 1.5a): the weight mg at the midpoint, the rail reaction S perpendicular to the rod, and the floor components F (horizontal) and V (vertical) at A.

A **rigid body** 刚体 in equilibrium needs zero resultant force AND zero resultant **moment** 力矩 about any point. Choose the pivot to eliminate unknowns (a smooth rail's single unknown reaction is why moments about the floor contact pay off). Resolve horizontally and vertically for the contact components; on a rough floor at **limiting equilibrium** 极限平衡 **friction** 摩擦力 $F = \mu R$ with F horizontal.

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Worked example – a rod against a smooth rail (WME02 June 2025 Q6). A uniform rod AB (mass m, length $2a$) rests with A on rough ground against a smooth rail at C, $AC = 1.5a$, inclination θ, the **normal reaction** 法向反力 S of the rail, perpendicular to the rod. Moments about A: $S(1.5a) = mg\,a\cos\theta$, so $S = \frac{2mg\cos\theta}{3}$. Vertically: $V = mg - S\cos\theta = \frac{mg}{3}(3 - 2\cos^2\theta)$. Limiting equilibrium with $\mu = \frac{4}{7}$: $F = S\sin\theta$, so $\frac{4}{7} \cdot \frac{mg}{3}(3 - 2\cos^2\theta) = \frac{2mg}{3}\cos\theta\sin\theta$. Dividing by $\cos^2\theta$ and using $\sec^2\theta = 1 + \tan^2\theta$: $12\tan^2\theta - 14\tan\theta + 4 = 0$, i.e. $6\tan^2\theta - 7\tan\theta + 2 = 0$.

Checklist before you call this unit done

- Integrate $a(t)$ to v and s with constants from the initial conditions; handle total distance via the turning point; differentiate $\mathbf{r}(t)$ for vector velocity and apply impulses as $\mathbf{I} = m\Delta\mathbf{v}$.
- Build COM tables (mass/area/length weighting) with signed distances for opposite-side pieces and negative holes; use the booklet's wire and lamina results; find suspension angles by trigonometry at the suspension point plus any isosceles-triangle angle.
- Apply work-energy and conservation of mechanical energy, with resistance work over slope distance and $P = Fv$ both ways.
- Solve direct collisions with momentum + restitution; state both speeds; compute the $(1 - e^2)$ energy loss and per-body impulses; chain wall rebounds and second collisions with a signed diagram.
- For rigid bodies: moments about the unknown-killing pivot, resolve for components, and close limiting-equilibrium problems into an equation in $\tan\theta$.

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Statistics 2

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Statistics 2: distributions, continuous variables and hypothesis tests

Statistics 2 (S2, paper WST02/01, 1 hour 30 minutes, 75 marks, first assessment June 2020) follows S1 in the applied route: P1–P2 and S1 are assumed. The booklet carries the binomial, Poisson and Normal tables. This reference covers the binomial and Poisson distributions with their approximations, continuous random variables, the uniform distribution, sampling and the full hypothesis-testing routine.

Must-know for S2 (not in the booklet):

Family	Formulae
binomial	$X \sim B(n,p)$: $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$
Poisson	$X \sim \mathrm{Po}(\lambda)$: $E(X) = \mathrm{Var}(X) = \lambda$; rates scale with time
continuous	$P(a<X<b) = \int_a^b f(x)\,dx$, $f(x) = F'(x)$, $E(X) = \int x f(x)\,dx$, $E(X^2) = \int x^2 f(x)\,dx$
uniform	on $[a,b]$: mean $\frac{a+b}{2}$, variance $\frac{(b-a)^2}{12}$, density $\frac{1}{b-a}$

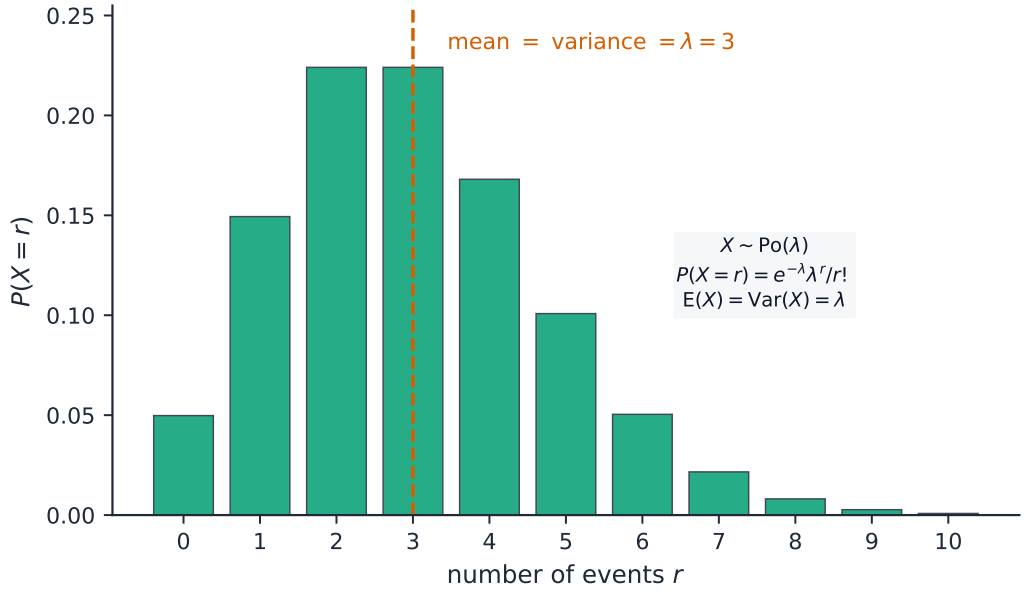
The binomial and Poisson distributions

Binomial distribution 二项分布: n independent trials, constant probability p – the two assumptions to state in context. **Poisson distribution** 泊松分布: events occur singly, independently, at a constant average rate – the rate scales ($\mathrm{Po}(\lambda)$ per minute gives $\mathrm{Po}(5\lambda)$ in 5 minutes). Moments come as formulas: np and $np(1-p)$; λ and λ. Given the mean and variance of a binomial, solve for n and p simultaneously. The **Poisson approximation** to the binomial needs n large and p small (use $\lambda = np$).

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People arriving at random at a steady average rate: the number per minute follows a Poisson distribution



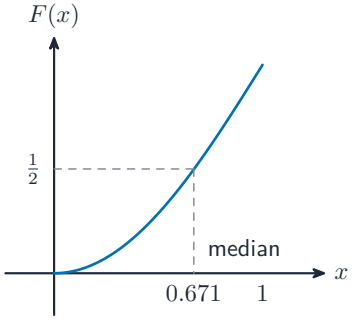
The Poisson distribution Po(3)*: for a Poisson variable the mean and variance both equal* λ*.*

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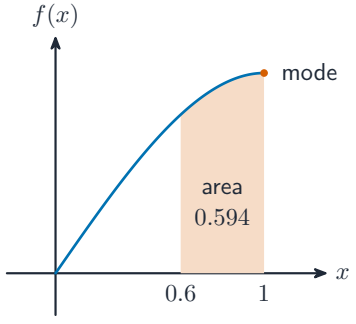
Worked example – moments then approximations (WST02 June 2025 Q3). Blue sweets per small bag have mean 3 and variance 2.55: $np = 3$ and $np(1-p) = 2.55$ give $1-p = 0.85$, so $p = 0.15$ and $n = 20$. A medium bag of 40 with 20% green: $X \sim B(40, 0.2)$, $E(X) = 8$, and $P(X > 8) = 1 - P(X \leq 8) = 0.407$. A large bag of 80 with 10% yellow: with $n = 80$ large and $p = 0.1$ small, $X \approx \text{Po}(8)$ and $P(X \geq 12) = 1 - P(X \leq 11) = 0.112$.

Continuous random variables

A **continuous random variable** has a **probability density function** 概率密度函数 f with total area 1 and $P(X = x) = 0$ at every point; the **cumulative distribution function** 分布函数 $F(x) = P(X \leq x)$ satisfies $f = F'$ and $F(a) = \int_{-\infty}^{a} f$. Piecewise densities: normalise $\int f = 1$ (which pins the unknown constant), then read off the **median** 中位数 ($F = \frac{1}{2}$), quartiles, and the **mode** 众数 from the peak of f (set $f' = 0$, or inspect corners). $E(X) = \int x f\,dx$, $E(X^2) = \int x^2 f\,dx$, Var $= E(X^2) - [E(X)]^2$; skewness compares mean, median and mode.



median solves $F(m) = \frac{1}{2}$;
mode differentiates $f = F'$

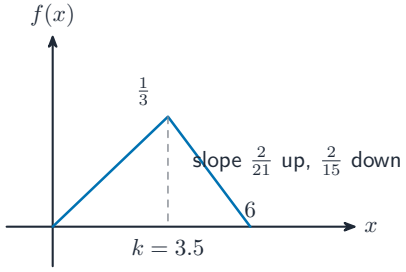


$P(X > 0.6) = 1 - F(0.6)$;
peak of f at the boundary

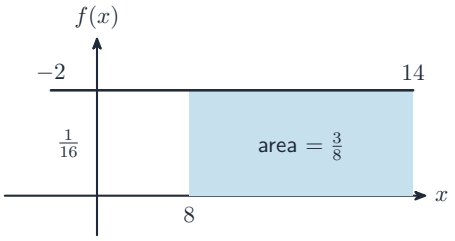
Left: the CDF F(x) = (6x^2 - x^4)/5 on [0, 1] rising from 0 to 1 with the median marked at F = 1/2 (x = 0.671). Right: the density f(x) = (12x - 4x^3)/5 with the mode at x = 1 and the area beyond x = 0.6 shaded (0.594).

Worked example – a CDF worked through (WST02 June 2025 Q1). $F(x) = k(6x^2 - x^4)$ on $[0, 1]$: $F(1) = 1$ gives $5k = 1$. $P(X > 0.6) = 1 - \frac{6(0.36) - 0.1296}{5} = 0.594$. The median: $\frac{6m^2 - m^4}{5} = \frac{1}{2}$, i.e. $2m^4 - 12m^2 + 5 = 0$, so $m = 0.671$. The mode: $f(x) = \frac{12x - 4x^3}{5}$ and $f'(x) = \frac{12 - 12x^2}{5} = 0$ gives $x = 1$ (reject -1, outside the support). Skewness: mean 0.64 < median 0.671 < mode 1 – negative skew.

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normalising pins k: $(2k-7)^2 = 0$, continuous at the peak



uniform: probabilities are lengths over $b - a = 16$

Left: a triangular pdf rising along 2x/21 from 0 to k = 3.5 then falling along 2(6-x)/15 to x = 6, peak value 1/3 marked. Right: the uniform density on [-2, 14] at height 1/16 with the interval [8, 14] shaded (area 3/8).

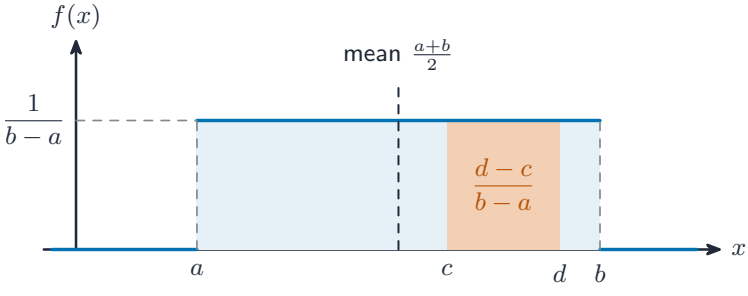
Worked example – a piecewise-linear pdf, no calculator (WST02 June 2025 Q5). With $f(x) = \frac{2x}{21}$ on $[0, k]$ and $f(x) = \frac{2(6-x)}{15}$ on $(k, 6]$: normalising,

$$\frac{k^2}{21} + \frac{2}{15}\left(18 - 6k + \frac{k^2}{2}\right) = 1 \implies 4k^2 - 28k + 49 = 0 \implies (2k-7)^2 = 0,$$

so $k = 3.5$ (the pdf is continuous there, both pieces equalling $\frac{1}{3}$). Then $E(X) = \frac{2}{21}\cdot\frac{k^3}{3} + \frac{2}{15}\int_{3.5}^{6}(6x - x^2)dx = \frac{19}{6}$, and with $E(X^2) = \frac{277}{24}$ given, $\mathrm{Var}(X) = \frac{277}{24} - \frac{361}{36} = \frac{109}{72}$.

The continuous uniform distribution

The **continuous uniform distribution** 均匀分布 on $[a, b]$ Uniform on $[a, b]$: density $\frac{1}{b-a}$, straight CDF, mean $\frac{a+b}{2}$, variance $\frac{(b-a)^2}{12}$. Two facts pin a and b (an interval probability gives $b - a$; the mean gives $a + b$). Coding scales the mean; probabilities are lengths over $b - a$.



probability = length $\div$ $(b - a)$

$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$

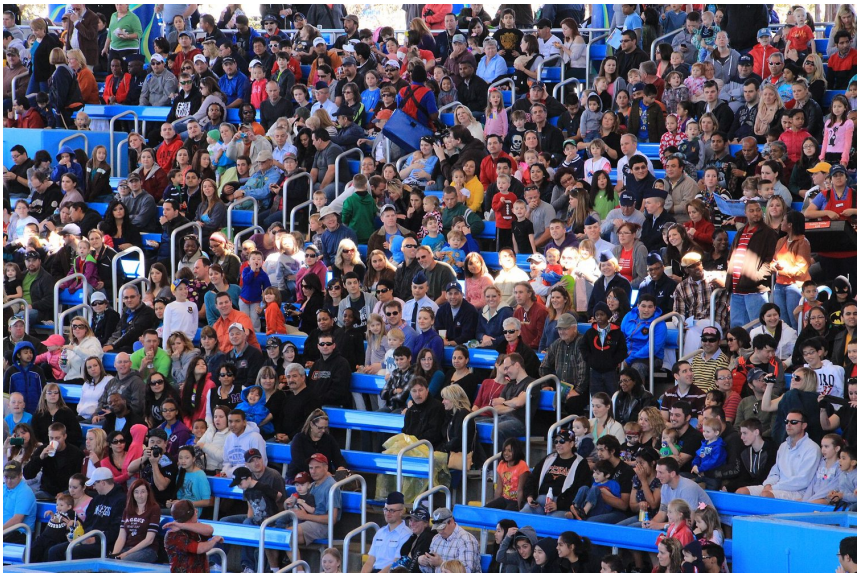
Uniform on [a, b]: probabilities are lengths divided by b − a

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Worked example – uniform on a wide interval (WST02 June 2025 Q6). Assume $a \leq 2$ and $b \geq 5$. Then $P(2 < X < 5) = \frac{3}{16}$ gives $b - a = 16$; $E(X) = 6$ gives $a + b = 12$: so $a = -2$, $b = 14$ Without that support assumption, $a = 4.4$, $b = 7.6$ also satisfies the two given facts; the following calculations use the stated support assumption. $E(cX + 1) = 3$ gives $c = \frac{1}{3}$; Var $= \frac{256}{12} = \frac{64}{3}$; $E(X^2) = \frac{64}{3} + 36 = \frac{172}{3}$; and $P(\frac{3}{2}X - b > a) = P(X > \frac{2(a+b)}{3}) = P(X > 8) = \frac{14-8}{16} = \frac{3}{8}$.

Populations, samples and statistics

A **census** 普查 aims to measure everybody (complete but costly, slow, possibly destructive). To **sample** 抽样 is to measure a selected part of the population; the selected units are also called a sample (cheaper and faster, but with sampling error). The **sampling frame** 抽样框 lists the sampling units. A **statistic** 统计量 is a function of the sample values only (no unknown parameters) – $\bar{X}$ is, $\bar{X} - \mu$ is not; its distribution over repeated samples is the **sampling distribution** 抽样分布.



Statistics studies a sample to learn about a whole population.

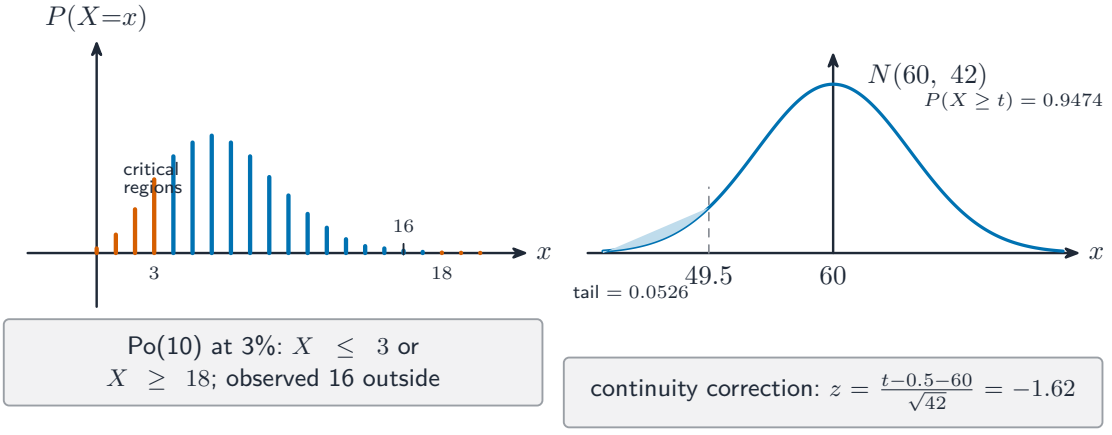
Worked example – a sampling distribution (WST02 June 2025 Q4). Three beads from a 2:5 black:white bag, scoring 4 per black and 1 per white: with $P(\text{black}) = \frac{2}{7}$, the score X takes 3, 6, 9, 12 with probabilities $\frac{125}{343}, \frac{150}{343}, \frac{60}{343}, \frac{8}{343}$ (binomial weights $\binom{3}{0}(\frac{5}{7})^3$ etc.). If Y counts 9-point sets among n independent plays, $Y \sim B(n, \frac{60}{343})$ and $1 - (\frac{283}{343})^n > 0.95$ needs $n > 15.6$, so $n = 16$.

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Hypothesis tests

A **hypothesis test** 假设检验 compares data against a null model.

State the **null hypothesis** 原假设 H_0 (no change, the quoted parameter) and H_1 (one- or two-tailed as asked). Find the **critical region** 临界域 at the stated nominal level by choosing tails whose total null probability does not exceed α; report the actual **significance level** 显著性水平 as the total tail probability. One-tailed tests put the whole level in one tail; two-tailed split it. Conclude IN CONTEXT: "insufficient evidence that … has changed". Normal approximations to the binomial: mean np, variance $np(1-p)$, with the **continuity correction** 连续性修正 (at least t means $t-0.5$).



Left: the Po(10) distribution as vertical bars with the critical regions X <= 3 and X >= 18 in a contrasting colour and the observed 16 marked between them. Right: the N(60, 42) curve with the area below 49.5 shaded and the continuity-corrected z = -1.62 marked.

Worked example – a Poisson rate test (WST02 June 2025 Q2). Customers average 20/hour; after alterations one 30-minute interval is watched: $X \sim \text{Po}(10)$, $H_0\colon \lambda = 10$, $H_1\colon \lambda \neq 10$. At 3%: $P(X \leq 3) = 0.0103$ but $P(X \leq 4) = 0.0293$ (too big); $P(X \geq 17) = 0.0270$ but $P(X \geq 18) = 0.0143$ – critical region $X \leq 3$ or $X \geq 18$, actual level $0.0103 + 0.0143 = 2.46\%$. The observed 16 lies outside → do not reject H_0.

Worked example – binomial test with a Normal middle (WST02 June 2025 Q7). 30% of customers take insurance; a sample of 40 gives $X \sim B(40, 0.3)$. Largest r with $P(X < r) < 0.05$ (fewer than r): $P(X \leq 6) = 0.0238 < 0.05$ but $P(X \leq 7) = 0.0553 > 0.05$, so $r = 7$. For 200 customers, $B(200, 0.3) \approx N(60, 42)$: $P(X \geq t) = 0.9474$ means $\frac{t-0.5-60}{\sqrt{42}} = -1.62$, so $t = 50$. After a special offer, 11 of 25 buy: $H_0\colon p = 0.3$, $H_1\colon p > 0.3$; $P(X \geq 11) = 0.0978 < 0.10$, so reject H_0 – evidence the proportion has increased.

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Checklist before you call this unit done

- State binomial/Poisson assumptions in context; solve mean/variance pairs for n, p; scale Poisson rates with time.
- Normalise piecewise densities (constants, perfect-square k); differentiate F for f; find median/quartiles from F, mode from f' or corners; run E(X), E(X^2), Var; read skewness from mean–median–mode.
- Quote and derive the uniform facts; pin a and b from an interval plus the mean; keep probabilities as lengths.
- Distinguish census vs sample with pros and cons; recognise statistics; build small sampling distributions by enumeration; wrap counts in B(n, p) for follow-ups.
- Run one- and two-tailed tests: hypotheses in symbols and words, critical region by accumulation, actual significance level, in-context conclusion; Normal approximation with continuity correction when n is large.

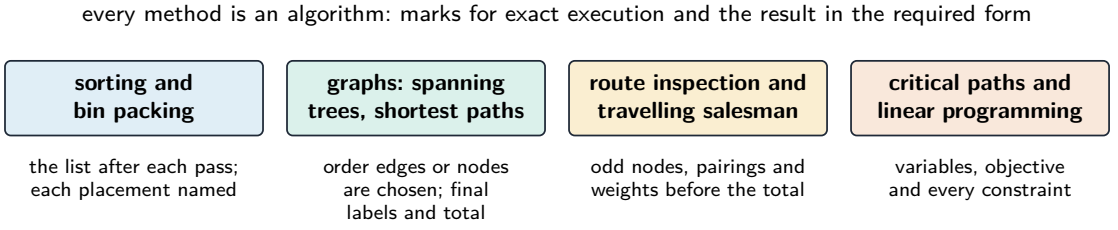
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Decision Mathematics 1

Pearson Edexcel · International A-Level · Mathematics

Decision Mathematics 1: algorithms, networks and linear programming

Decision Mathematics 1 (D1, paper WDM01/01, 1 hour 30 minutes, 75 marks, first assessment June 2019) is the discrete option of the IAL: P1–P2 are assumed, and every method is an ALGORITHM – a finite list of exact instructions. No formula booklet entries beyond the basics: the marks are for executing procedures precisely and stating results in the required form. This reference covers sorting and packing algorithms, minimum spanning trees, shortest paths, the Chinese postman and travelling salesman problems, critical path analysis and linear programming.



Every D1 method is an algorithm: the marks are for exact steps in the required form

House rules for algorithm answers:

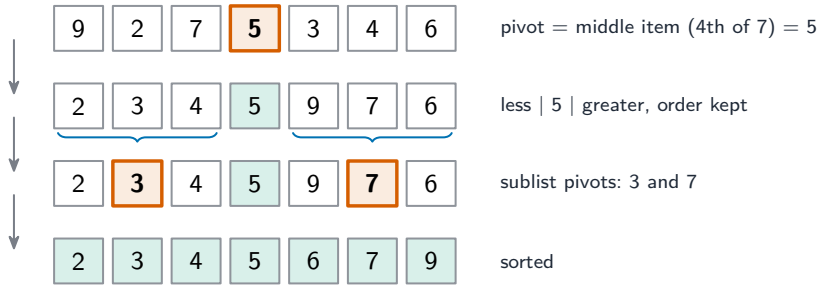
Task	Required form
sorting	list the list after EACH pass (or pivot choices for quick sort)
bin packing	name the algorithm (first-fit, first-fit decreasing, full-bin) and show each placement
graph algorithms	state the order edges/nodes are chosen; show final values (labels, total weight)
route inspection	list the odd nodes, the pairings and their weights, before totalling
linear programming	define variables, objective and EVERY constraint with non-negativity

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Sorting and packing algorithms

The **bubble sort** 冒泡排序 compares adjacent items, swapping out-of-order pairs, one pass at a time; a pass with no swaps ends it. The **quick sort** 快速排序 picks the MIDDLE item of the list as pivot, partitions the remaining items into "less than pivot", "equal to pivot" and "greater than pivot" groups (keeping every copy); it recursively sorts the smaller and larger groups – always quote the pivot and the list after each partition. The spec pins the middle-item choice to the method defined in its glossary, so exercise lists here keep an odd number of items and the middle item is the one at position $\frac{n+1}{2}$. **Bin packing** 装箱问题: first-fit places each item in the first bin with room; first-fit decreasing sorts items largest-first then applies first-fit; full-bin packing looks for combinations filling a bin exactly.

> **Worked example – quick sort (original).** Sort 9, 2, 7, 5, 3, 4, 6. The middle item (position 4 of 7) is 5: less = 2, 3, 4; greater = 9, 7, 6, giving 2, 3, 4, **5**, 9, 7, 6. Sublist 2, 3, 4 (pivot 3): 2, **3**, 4 – sorted. Sublist 9, 7, 6 (pivot 7): 6, **7**, 9 – sorted. Final list: 2, 3, 4, 5, 6, 7, 9.

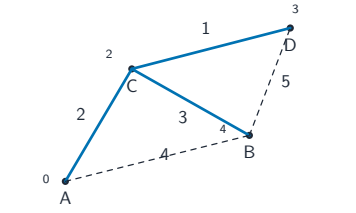


Quick sort with the middle-item pivot: quote the pivot and the list after every partition

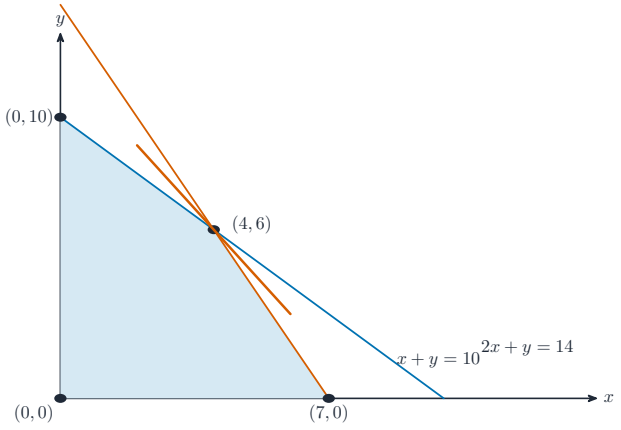
> **Worked example – bin packing (original).** Bins of capacity 10; items 7, 5, 4, 3, 6, 2 (total 27, so at least 3 bins). First-fit: 7→b1, 5→b2 (7+5 > 10), 4→b2 (9), 3→b1 (10), 6→b3, 2→b3 (8): bins (7,3), (5,4), (6,2). First-fit decreasing: 7, 6, 5, 4, 3, 2: 7→b1, 6→b2, 5→b3 (6+5 > 10), 4→b2 (10), 3→b1 (10), 2→b3 (7): bins (7,3), (6,4), (5,2) – also 3 bins.

Minimum spanning trees

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MST = AC + CD + CB = 6; Dijkstra: A 0, C 2, B 4, D 3 via A–C–D



$P = 3x + 2y$: vertices give 0, 21, 24, 20 — max at $(4, 6)$

Left: a weighted network on A, B, C, D with the MST edges AC 2, CD 1, CB 3 highlighted and Dijkstra labels 0, 2, 3, 8 at the vertices. Right: the feasible region of a linear program with vertices (0,0), (7,0), (4,6), (0,10) and the objective line touching the optimum at (4,6).

A **minimum spanning tree (MST)** 最小生成树 connects every vertex with the least total weight. **Kruskal's algorithm** 克鲁斯卡尔算法: sort edges by weight, add the smallest that does NOT form a cycle (reject and move on when it would). **Prim's algorithm** 普里姆算法: grow a tree from any start vertex, repeatedly adding the cheapest edge from the tree to a new vertex. The matrix form of Prim: cross out the start row, ring the cheapest remaining entry in the column of the chosen vertex, and continue. Kruskal is edge-driven (good on a drawn network); Prim is vertex-driven (good on a matrix). Both give the same total; edges may differ only when weights tie.

> **Worked example (original).** Edges AB 4, AC 2, BC 3, BD 5, CD 1: Kruskal takes CD 1, AC 2, BC 3 (B joins – no cycle), then rejects AB 4 and BD 5 (cycles): MST total 6. Prim from A: AC 2, CD 1 (cheapest edge from {A,C} to a new vertex), CB 3: the same tree, and the total 6 ✓. Always list the edges in the order chosen and give the total.

Shortest paths

For non-negative edge weights, **Dijkstra's algorithm** 迪杰斯特拉算法 labels vertices with the smallest total distance found so far: start vertex label 0; repeatedly pick the unlabelled vertex of smallest temporary label, make it permanent, and update its neighbours (new = permanent + weight, only if smaller). Record the order of permanence and the final labels; the route is recovered by tracing back through vertices whose label equals the final label minus the connecting weight.



A metro map is a network: Dijkstra's algorithm finds the shortest route between two stations

> **Worked example (original).** Edges AB 4, AC 2, BC 1, BD 5, CD 8. Labels: A = 0 permanent; C = 2 permanent; B = min(4, 2+1) = 3 permanent; D = min(3+5, 2+8) = 8 permanent. The shortest route from A to D has length 8: A–C–B–D (trace back: 8 = 3 + 5 at B, 3 = 2 + 1 at C).

The Chinese postman problem

The **route inspection (Chinese postman) problem** 中国邮递员问题 walks every EDGE at least once and returns to the start. Total length = total of all edges + the shortest way to duplicate edges so every vertex becomes even. Find the odd vertices (at most four), list their pairings, choose the pairing of least total weight, and duplicate those edges. State the pairing before the total.

> **Worked example (original).** Edges AB 4, AC 2, BC 3, BD 5, CD 1 (total 15). Degrees: A 2, B 3, C 3, D 2 – the odd nodes are B and C, so duplicate BC (weight 3): the shortest closed route walking every edge at least once has length 15 + 3 = 18.

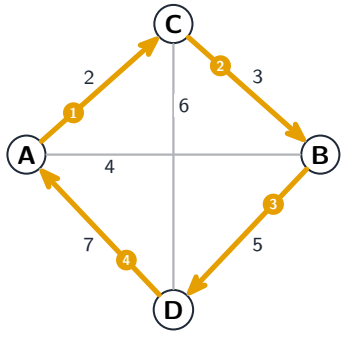
The travelling salesman problem

The **classical travelling salesman problem** 旅行商问题 tours every VERTEX exactly once and returns, on a complete graph with the triangle inequality. Bounds: the MST of the network doubled gives an UPPER bound route (walk each MST edge twice); a better upper bound applies shortcuts. Lower bounds come from MST-based arguments (e.g. the MST itself, or deleting a vertex and adding its two cheapest edges). The **nearest neighbour algorithm** 最近邻算法 builds a tour: from the

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start vertex repeatedly go to the nearest unvisited vertex, then return. Convert to a complete network of shortest distances first when the graph is not complete (via Dijkstra or inspection).

> **Worked example (original).** Complete network: AB 4, AC 2, AD 7, BC 3, BD 5, CD 6 (the triangle inequality holds throughout). Nearest neighbour from A: A–C (2), C–B (3), B–D (5), D–A (7): tour length 17 – an upper bound. The MST is AC + BC + BD = 10, so doubling it gives the crude upper bound 20, improved to 17 by shortcuts; a lower bound from deleting A (MST of B, C, D is $3 + 5 = 8$, plus A's two cheapest edges AC + AB $= 2 + 4 = 6$) is 14. The optimal tour lies between 14 and 17.



nearest neighbour from A:
A–C–B–D–A $= 2 + 3 + 5 + 7 = 17$
upper bound
delete A: MST of B, C, D $= 3 + 5 = 8$
plus AC + AB $= 2 + 4$: lower bound 14
$14 \leq$ optimal tour ≤ 17

Nearest neighbour gives an upper bound, the deleted-vertex MST a lower bound: the optimal tour lies between 14 and 17

Critical path analysis

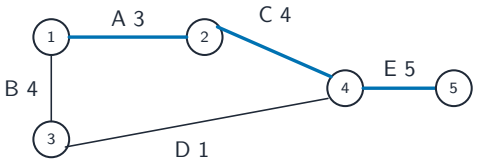
Model a project as an **activity-on-arc network** 活动在弧上的网络 from a precedence table; **dummies** 虚活动 (zero-weight arcs) resolve dependencies that shared endpoints cannot express. The **critical path** 关键路径 is the longest chain through the network: the project duration. Compute **earliest event times** 最早事件时间 (forward pass: the max of arrivals) and **latest event times** 最晚事件时间 (backward pass: the min of departures). An activity's **total float** 总时差 = latest finish − earliest start − duration; critical activities have zero float. **Gantt (cascade) charts** 甘特图 bar each activity from its earliest start with its float as a trailing space; scheduling allocates workers day by day.

> **Worked example (original).** Activities A (3 days), B (4), C (4 after A), D (1 after B), E (5 after C and D). Chain A–C–E = 12 days; chain B–D–E = 10 days: the critical path is A–C–E and the project takes 12 days; D's total float = 12 − (4 + 1 + 5) = 2 days.

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first-fit: (7,3), (5,4), (6,2) — three bins, none over capacity



event times 0; 3 / 4; 7; 12

critical path A–C–E = 12 days; D has float 2

Left: three first-fit bins of capacity 10 holding items (7,3), (5,4), (6,2). Right: an activity-on-arc network with events 1-5, activities A 3, B 4, C 4, D 1 and E 5, the critical arcs highlighted, and event times 0, 3/4, 7, 12.

Linear programming

Formulate: define the decision variables, write the objective (maximise/minimise) and every constraint as an inequality, including $x, y \geq 0$. Solve two-variable problems graphically: shade the feasible region, then use the **ruler method** 尺子法 (slide a ruler along the objective's gradient) or the **vertex method** 顶点法 (evaluate the objective at each vertex of the feasible region). Integer solutions: test integer points near the optimal vertex inside the region.

Checklist before you call this unit done

- Execute bubble and quick sorts showing every pass and pivot; bin pack by first-fit, first-fit decreasing and full-bin with placements shown.
- Find MSTs by Kruskal and by Prim (node and matrix forms); draw a network from a matrix and back.
- Run Dijkstra with labels in order of permanence; recover the route, not just the length.
- Solve route inspection: odd nodes, all pairings with weights, least duplication, total walk length.
- Bound the classical TSP: doubled MST and shortcuts for upper bounds, MST-deletion for lower bounds; run nearest neighbour on the complete shortest-distance network.
- Build activity networks with dummies; forward and backward passes for event times; total floats; identify the critical path; draw the Gantt chart and schedule.

> **Worked example – linear programming (original).** Maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $2x + y \leq 14$, $x, y \geq 0$. The feasible region's vertices are $(0, 0)$, $(7, 0)$, $(4, 6)$, $(0, 10)$; evaluating P: 0, 21, 24, 20 – the maximum is $P = 24$ at $(4, 6)$, where both constraints are tight.

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- Formulate linear programs fully; solve graphically by ruler or vertex methods; find best integer solutions.

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