4 · Pure Mathematics 4

Pearson Edexcel · International A-Level · Mathematics

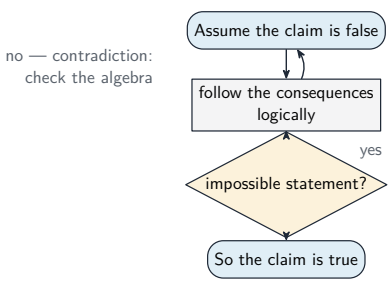
Pure Mathematics 4: proof, parametrics and the full calculus

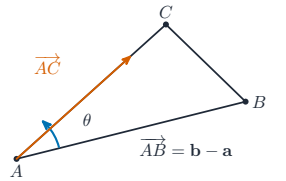
Pure Mathematics 4 (P4) is the final compulsory pure unit of International A Level Mathematics and International A Level Pure Mathematics. P1–P3 are assumed and may be tested on the WMA14/01 paper (1 hour 30 minutes, 75 marks; same calculator and formula-booklet rules). This reference covers proof by contradiction, partial fractions, parametric equations, the general binomial series, implicit and parametric differentiation, differential equations, the extended integration methods, volumes of revolution, and vectors.

Must-know for P4 (not in the booklet):

Family	Formulae
binomial series	$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$, valid $\lvert x\rvert < 1$ for any rational n
volume of revolution	$V = \pi \int_a^b y^2\,dx$ about the x-axis (parametric: $\pi \int y^2 \frac{dx}{dt}\,dt$)
vectors	column form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; $\lvert\mathbf{a}\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$; unit vector $\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}$; $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Proof by contradiction





Angle from the scalar product: $\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\lvert AB\rvert\lvert AC\rvert}$

$\sqrt{2}$ irrational and the infinity of primes both run this loop

Proof by contradiction assumes the opposite of the claim, derives an impossibility, and concludes the original claim. Two classics must be known:

Worked example – $\sqrt{2}$ is irrational. (A number is **irrational** when no fraction $\frac{p}{q}$ equals it.) Suppose instead $\sqrt{2} = \frac{p}{q}$ in lowest terms (p, q coprime integers, $q \neq 0$).

- Squaring: $2 = \frac{p^2}{q^2} \implies p^2 = 2q^2$, so p^2 is even, so p is even: write $p = 2r$.
- Then $4r^2 = 2q^2 \implies q^2 = 2r^2$, so q is even too.
- Both p and q even contradicts $\frac{p}{q}$ being in lowest terms. ■

Worked example – there are infinitely many primes. Suppose finitely many: $p_1, \ldots, p_k$.

- Consider $N = p_1 p_2 \cdots p_k + 1$. Dividing N by any p_i leaves remainder 1.
- So N is either prime itself or has a prime factor not in the list – either way a prime outside $\{p_1, \ldots, p_k\}$, contradicting the assumption. ■

Worked example – a trigonometric contradiction (adapted from Edexcel IAL WMA14 June 2025 Q10). Prove that $\lvert\sin x + \cos x\rvert < 1$ for all $90° < x < 180°$.

- Suppose not: some x in the range has $\lvert\sin x + \cos x\rvert \geq 1$. Squaring (both sides non-negative):

$$(\sin x + \cos x)^2 \geq 1 \implies \sin^2 x + 2\sin x\cos x + \cos^2 x \geq 1 \implies 1 + \sin 2x \geq 1 \implies \sin 2x \geq 0$$

- But $90° < x < 180°$ gives $180° < 2x < 360°$, where $\sin 2x < 0$ – a contradiction.
- So no such x exists and $\lvert\sin x + \cos x\rvert < 1$ throughout the range. ■
- Check at $x = 135°$: $\lvert\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\rvert = 0 < 1$ ✓; at $x = 100°$: $\lvert 0.985 - 0.174\rvert = 0.811 < 1$ ✓.

Partial fractions

Decompose rational functions into **partial fractions**. Denominators stay at repeated linear complexity: $(ax+b)(cx+d)(ex+f)$ or $(ax+b)(cx+d)^2$ with the **repeated factor** $(cx+d)$ split across powers; quadratic factors are not required. **Improper fractions** (numerator degree $\geq$ denominator's) divide first.

Worked example – distinct factors. Decompose $\frac{5x+1}{(x+2)(x-1)}$ (from the P2 sheet family):

$$\frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$$

- Check at $x = 0$: LHS $= \frac{1}{-2} = -\frac{1}{2}$; RHS $= \frac{3}{2} - 2 = -\frac{1}{2}$ ✓.

Worked example – repeated factor. Decompose $\frac{3x^2+7x+6}{x(x+1)^2}$.

- Set up with the repeated factor split across both powers:

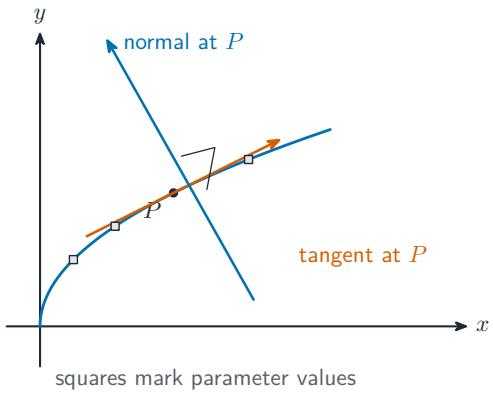
$$\frac{3x^2+7x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

- Multiply through: $3x^2 + 7x + 6 = A(x+1)^2 + Bx(x+1) + Cx$.
- At $x = 0$: $6 = A$. At $x = -1$: $3 - 7 + 6 = 2 = -C \Rightarrow C = -2$.
- Compare x^2: $3 = A + B \Rightarrow B = -3$.

$$\frac{3x^2 + 7x + 6}{x(x+1)^2} = \frac{6}{x} - \frac{3}{x+1} - \frac{2}{(x+1)^2}$$

- Check at $x = 1$: LHS $= \frac{16}{4} = 4$; RHS $= 6 - \frac{3}{2} - \frac{1}{2} = 4$ ✓.

Parametric equations



Parametric gradient: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$;
the normal takes its negative reciprocal.

Parametric equations give both coordinates through a parameter t: $x = f(t)$, $y = g(t)$. Convert to **Cartesian** form by making t the subject of one equation and substituting into the other (or by exploiting an identity). Never forget the excluded values of x where the parameter map breaks.

> **Worked example.** A curve has $x = 2t$, $y = t^2 - 1$. Find the Cartesian equation.

- From $x = 2t$: $t = \frac{x}{2}$, so

$$y = \left(\frac{x}{2}\right)^2 - 1 = \frac{x^2}{4} - 1$$

- Check at $t = 3$: $(x, y) = (6, 8)$; Cartesian: $36/4 - 1 = 8$ ✓.

Worked example – parametric point and normal (verified facts from Edexcel IAL WMA14 June 2025 Q5). A parametric curve passes through $P(-7, -3)$ at $t = 2$ with $\frac{dy}{dx} = -\frac{4}{5}$ there (from $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$).

- Normal gradient: $-\frac{1}{-4/5} = \frac{5}{4}$.

$$y + 3 = \tfrac{5}{4}(x + 7) \implies 5x - 4y + 23 = 0$$

- Check: $5(-7) - 4(-3) + 23 = -35 + 12 + 23 = 0$ ✓.

The binomial series for any rational n

For any **rational** n and $|x| < 1$:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Factor first: $(a+bx)^n = a^n\left(1 + \frac{b}{a}x\right)^n$, valid for $|x| < \left|\frac{a}{b}\right|$. Decompose rational functions into partial fractions, then expand each piece. Powers do not terminate unless n is a positive integer.

Worked example (adapted from Edexcel IAL WMA14 June 2025 Q8). Expand $\sqrt{4+x}$ to three non-zero terms.

- Factor: $\sqrt{4+x} = 2\left(1 + \frac{x}{4}\right)^{1/2}$, valid $|x| < 4$, with $n = \frac{1}{2}$:

$$\left(1 + \frac{x}{4}\right)^{1/2} = 1 + \frac{1}{2} \cdot \frac{x}{4} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{x}{4}\right)^2 + \cdots = 1 + \frac{x}{8} - \frac{x^2}{128} + \cdots$$

$$\sqrt{4+x} = 2 + \frac{x}{4} - \frac{x^2}{64} + \cdots$$

- Check at $x = 1$: series gives $2 + 0.25 - 0.0156 = 2.234$; true $\sqrt{5} = 2.2361\ldots$ ✓ (two terms already accurate to 2 d.p.).

Implicit and parametric differentiation

Implicit differentiation differentiates both sides with respect to x, treating y as a function of x: $\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$, and products such as xy need the product rule: $\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$. Then solve for $\frac{dy}{dx}$.

Parametric differentiation: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Worked example – an implicit tangent (adapted from Edexcel IAL WMA14 June 2025 Q1). The curve $2y^2 - 6xy + 7e^{1-2x} = 27$ passes through the point $\left(\frac{1}{2}, 4\right)$. Find the tangent there.

- Differentiate term by term:

$$4y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx} - 14e^{1-2x} = 0 \implies \frac{dy}{dx} = \frac{6y + 14e^{1-2x}}{4y - 6x}$$

- At $\left(\frac{1}{2}, 4\right)$: $e^{1-1} = 1$, so
$$\frac{dy}{dx} = \frac{24+14}{16-3} = \frac{38}{13}$$
- Tangent: $y - 4 = \frac{38}{13}\left(x - \frac{1}{2}\right) \implies 38x - 13y + 33 = 0$.
- Check: $38\left(\frac{1}{2}\right) - 13(4) + 33 = 19 - 52 + 33 = 0$ ✓.

Differential equations and connected rates

Form an equation from a rate description, then solve **separable** equations:

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\,dx$$

Use a boundary condition for the **particular solution**. **Connected rates of change** use the chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$.

Worked example – a melting sphere (Edexcel IAL WMA14 June 2025 Q2). A sphere of radius r cm has volume $V = \frac{4}{3}\pi r^3$ and melts with $\frac{dV}{dt} = k$ (constant, in cm^3 per second). Show $\frac{dr}{dt} \propto \frac{1}{r^2}$.

- Differentiate the volume link: $\frac{dV}{dr} = 4\pi r^2$.
- Chain rule: $k = \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, so
$$\frac{dr}{dt} = \frac{k}{4\pi r^2} = \frac{k}{4\pi} \cdot \frac{1}{r^2}$$
- The constant $\frac{k}{4\pi}$ multiplies $\frac{1}{r^2}$: the rate of decrease of the radius is inversely proportional to r^2. ■
- Interpretation: small spheres vanish disproportionately fast – the last millimetre melts quickest.

Worked example – a separable equation (adapted from Edexcel IAL WMA14 June 2025 Q3). Solve $y\frac{dy}{dx} = 3\sec 2x \tan 2x$, given $y = 4$ at $x = \frac{\pi}{6}$.

- Separate: $\int y\,dy = \int 3\sec 2x \tan 2x\,dx$ (the right side integrates to $\frac{3}{2}\sec 2x$, reversing $\frac{d}{dx}\sec 2x = 2\sec 2x \tan 2x$):
$$\frac{y^2}{2} = \frac{3}{2}\sec 2x + c \implies y^2 = 3\sec 2x + C$$

- Substitute $\left(\frac{\pi}{6}, 4\right)$: $16 = 3\sec\frac{\pi}{3} + C = 3(2) + C$, so $C = 10$:
$$y^2 = 3\sec 2x + 10$$
- Check: $3(2) + 10 = 16 = 4^2$ ✓.

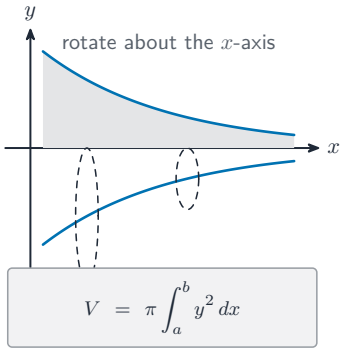
Integration: substitution, parts and partial fractions

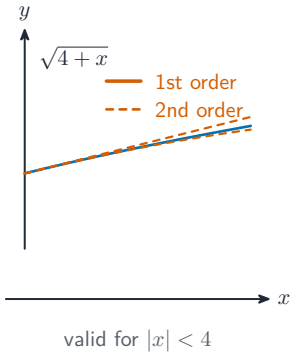
- **Substitution** (usually given): rewrite the integral in u, adjust dx via $\frac{du}{dx}$, integrate, substitute back. It is the chain rule reversed.
- **Integration by parts**: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ – the product rule reversed. Know $\int \ln x\,dx = x\ln x - x + c$ and repeated parts for $\int x^2 e^x dx$-type integrals.
- **Partial fractions** turn a rational integrand into sums of $\frac{A}{ax+b}$ (integrating to logs) and $\frac{C}{(ax+b)^2}$ (to powers).

Worked example – parts with a repeat in sight. Find $\int e^x \cos x\,dx$ (needed for volumes of revolution with trig curves).

- Parts with $u = e^x$, $\frac{dv}{dx} = \cos x$:
$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$
- Parts again on the new integral ($u = e^x$, $\frac{dv}{dx} = \sin x$):
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$
- The original integral reappears; collect it:
$$2\int e^x \cos x\,dx = e^x(\sin x + \cos x) + c \implies \int e^x \cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + c$$
- Check by differentiating: $\frac{1}{2}e^x(\sin x + \cos x) + \frac{1}{2}e^x(\cos x - \sin x) = e^x \cos x$ ✓.

Volumes of revolution and parametric areas





Rotating the region under $y = f(x) \geq 0$ about the x-axis sweeps a solid whose volume is

the **volume of revolution**

$$V = \pi \int_a^b y^2\,dx$$

Square the function **before** integrating, use identities to handle $\sin^2/\cos^2$, and split by linearity. With parametric equations, replace $y^2\,dx$ by $y^2 \dfrac{dx}{dt}\,dt$ between parameter values; areas under parametric curves use $y\dfrac{dx}{dt}\,dt$.

Worked example – volume with an identity (structure adapted from Edexcel IAL WMA14 June 2025 Q9). The region under $y = 3\cos x$ for $0 \leq x \leq \frac{\pi}{2}$ rotates about the x-axis; find the exact volume.

$$V = \pi \int_0^{\pi/2} 9\cos^2 x\,dx = 9\pi \int_0^{\pi/2} \tfrac{1}{2}(1+\cos 2x)\,dx = \frac{9\pi}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{9\pi}{2}\cdot\frac{\pi}{2} = \frac{9\pi^2}{4}$$

- Check: the average of $\cos^2 x$ over a half-turn is $\frac{1}{2}$, so $V \approx 9\pi \times \frac{1}{2} \times \frac{\pi}{2} = \frac{9\pi^2}{4}$ ✓ exactly.

Vectors

A **vector** has magnitude and direction, written $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ or as a column. The **magnitude** is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$; the **unit vector** in the direction of $\mathbf{a}$ is $\mathbf{a}/|\mathbf{a}|$.

Addition and scalar multiplication have geometric meanings (translations, scaling, **parallel** when one is a multiple of the other).

The **scalar product** $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ gives angles through $\cos\theta$. The **position vector** of B relative to A is $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Worked example – a 45° triangle (Edexcel IAL WMA14 June 2025 Q6, fully verified). $A = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $B = 5\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $C = 3\mathbf{i} + \alpha\mathbf{j} + 5\mathbf{k}$, and angle $BAC = 45°$.

- $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$, so $|\overrightarrow{AB}| = \sqrt{32}$; $\overrightarrow{AC} = \begin{pmatrix} 2 \\ \alpha - 2 \\ 8 \end{pmatrix}$.
- Scalar product: $\overrightarrow{AB}\cdot\overrightarrow{AC} = 8 + 0 + 32 = 40$.
- Angle condition with $\cos 45° = \frac{1}{\sqrt{2}}$:

$$\frac{40}{\sqrt{32}\sqrt{68 + (\alpha-2)^2}} = \frac{1}{\sqrt{2}} \implies \sqrt{68 + (\alpha-2)^2} = \frac{40\sqrt{2}}{\sqrt{32}} = \frac{40\sqrt{2}}{4\sqrt{2}} = 10$$

$$(\alpha - 2)^2 = 100 - 68 = 32 \implies \alpha = 2 \pm 4\sqrt{2}$$

- Check: $|\overrightarrow{AB}||\overrightarrow{AC}|\cos 45° = \sqrt{32} \times 10 \times \frac{1}{\sqrt{2}} = 4\sqrt{2} \times 10 \times \frac{1}{\sqrt{2}} = 40$ ✓ matches the scalar product.

Checklist before you call this unit done

- Structure contradiction proofs ($\sqrt{2}$, primes) with the assumption stated and the contradiction named.
- Decompose into partial fractions including repeated linear factors; divide improper fractions first; verify by a substituted value.
- Convert parametric ↔ Cartesian with excluded values; find parametric tangents and normals via dy/dx = (dy/dt)/(dx/dt).
- Expand $(a + bx)^n$ for rational n with the validity condition; combine expansions of partial fractions.
- Differentiate implicitly (product rule on mixed terms) and parametrically; form differential equations from connected rates.
- Solve separable equations with a boundary condition; check the particular solution.
- Integrate by substitution, parts (including repeats, $\ln x$), and partial fractions.
- Compute volumes of revolution $\pi \int y^2 dx$ and parametric areas via dx/dt.
- Use vectors: magnitude, unit vectors, position vectors; scalar-product angles.