3 · Pure Mathematics 3

Pearson Edexcel · International A-Level · Mathematics

Pure Mathematics 3: functions, identities and calculus at speed

Pure Mathematics 3 (P3) extends P1 and P2: rational functions and inverses, the modulus, combined transformations, reciprocal trigonometric functions and harmonic form, log-graph model fitting, the full differentiation rules, integration by recognition, and numerical methods. P1 and P2 knowledge is assumed and may be tested on the WMA13/01 paper (1 hour 30 minutes, 75 marks; same calculator and formula-booklet rules).

Formulae you must know for P3 (not in the booklet):

Family	Formulae
compound angle	$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$; $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$; $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
double angle	$\sin 2A \equiv 2 \sin A \cos A$; $\cos 2A \equiv \cos^2 A - \sin^2 A$; $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$
reciprocal identities	$\sec^2 A \equiv 1 + \tan^2 A$; $\operatorname{cosec}^2 A \equiv 1 + \cot^2 A$
derivatives	$\sin kx \to k \cos kx$; $\cos kx \to -k \sin kx$; $\tan kx \to k \sec^2 kx$; $e^{kx} \to ke^{kx}$; $\ln x \to \frac{1}{x}$; $a^x \to a^x \ln a$
integrals	$\int \cos kx = \frac{1}{k} \sin kx$; $\int \sin kx = -\frac{1}{k} \cos kx$; $\int e^{kx} = \frac{1}{k} e^{kx}$; $\int \frac{1}{x} = \ln\lvert x \rvert$

Algebra: rational expressions and functions

Simplify rational expressions by factorising and cancelling; **algebraic division** serves denominators up to quadratics. Only cancel factors, never terms.

A **function** is a one-one or many-one mapping; its **domain** is the allowed inputs and its **range** the outputs. The **composition** fg means "do g first, then f". The **inverse** f^{-1} undoes f (it exists where f is one-one), and $f^{-1}f(x) = ff^{-1}(x) = x$; the graph of f^{-1} reflects the graph of f in $y = x$.

Worked example – inverse and composition (adapted from Edexcel IAL WMA13 June 2025 Q1). $f(x) = \dfrac{2x}{3x+1}$, $x \geq 0$.

(a) Range of f.

- At $x = 0$: $f(0) = 0$; as $x \to \infty$: $f(x) \to \frac{2}{3}$ from below, and f is increasing (larger x gives larger $f(x)$).

$$0 \le f(x) < \tfrac{2}{3}$$

(b) Find $f^{-1}(x)$.

- Let $y = \dfrac{2x}{3x+1}$ and solve for x:

$$y(3x+1) = 2x \implies 3xy + y = 2x \implies x(3y-2) = -y \implies x = \frac{y}{2-3y}$$

$$f^{-1}(x) = \frac{x}{2-3x}$$

- Check: $f\left(\frac{x}{2-3x}\right) = \dfrac{2x/(2-3x)}{3x/(2-3x)+1} = \dfrac{2x}{3x+2-3x} = x$ ✓.

(c) Solve $f^{-1}(x) = f(x)$.

$$\frac{x}{2-3x} = \frac{2x}{3x+1} \implies x(3x+1) = 2x(2-3x) \implies x\big(3x+1-4+6x\big) = 0$$

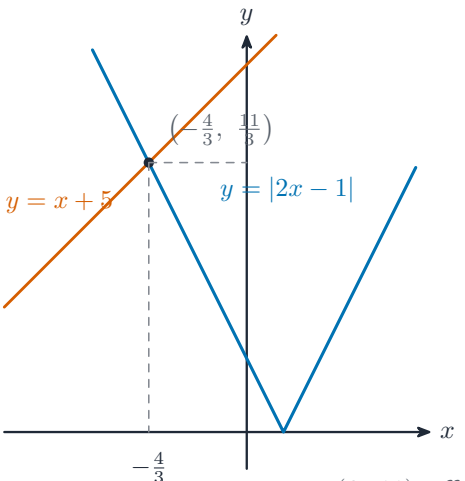
$$x(9x-3) = 0 \implies x = 0 \text{ or } x = \tfrac{1}{3}$$

- Check: $f(\frac{1}{3}) = \frac{2/3}{2} = \frac{1}{3} = f^{-1}(\frac{1}{3})$ ✓.

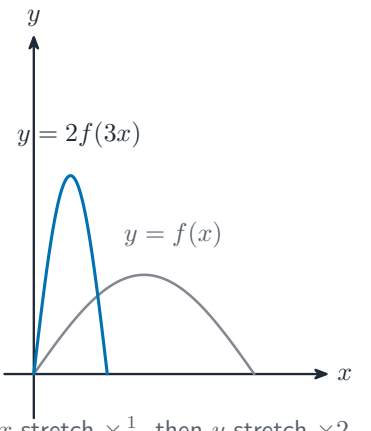
The modulus function and combined transformations

The **modulus** $|x|$ is x for $x \ge 0$ and $-x$ for $x < 0$: $y = |ax+b|$ is the line $y = ax+b$ with its negative part reflected above the axis. Given $y = f(x)$, sketch $y = |f(x)|$ (reflect any below-axis part) and $y = f(|x|)$ (keep the right half, reflect it left). Solve modulus equations graphically or by cases, testing each solution.

Combined transformations apply the P1 rules together, for example $y = 2f(3x)$ or $y = f(-x)+1$: stretches multiply, translations then shift; the order given in the syllabus is $y = af(x)$, $y = f(x)+a$, $y = f(x+a)$, $y = f(ax)$ combined in one expression ($y = f(ax+b)$ as such is not required).



second intersection at $(6,\ 11)$, off-screen



x-stretch $\times\frac{1}{3}$, then y-stretch $\times 2$

The V meets the line at the marked point; the second intersection (6, 11) lies off-screen. Combine transformations one factor at a time.

Worked example – modulus equation (specification's own example). Solve $|2x - 1| = x + 5$.

- Case $2x - 1 \geq 0$ ($x \geq \frac{1}{2}$): $2x - 1 = x + 5 \implies x = 6$ ✓ (satisfies $x \geq \frac{1}{2}$).
- Case $2x - 1 < 0$: $-(2x - 1) = x + 5 \implies -3x = 4 \implies x = -\frac{4}{3}$ ✓ ($< \frac{1}{2}$).
- Both solutions are $x = 6$ and $x = -\frac{4}{3}$. Check: $|2(6) - 1| = 11 = 6 + 5$ ✓; $|-\frac{8}{3} - 1| = \frac{11}{3} = -\frac{4}{3} + 5$ ✓.

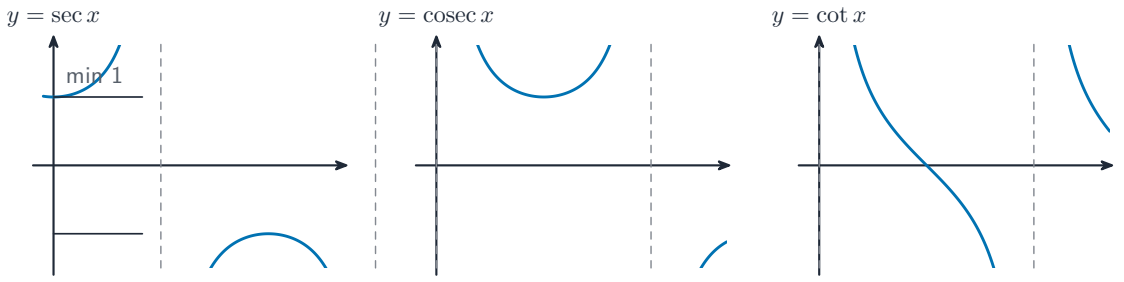
Further trigonometry: sec, cosec, cot and harmonic form

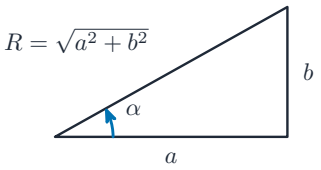
The **reciprocal functions** :

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

with identities $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$. Their graphs have **asymptotes** where the denominator function is zero. The **inverse functions** arcsin, arccos, arctan need **restricted domains** : $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arccos : [-1, 1] \to [0, \pi]$, $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.

Harmonic form : $a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$ with $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \dfrac{b}{a}$ – one number, one angle, ready to maximise or solve.





$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$ with $\tan\alpha = \dfrac{b}{a}$: one amplitude R and one phase shift α replace both terms.

Harmonic form packs $a\cos\theta + b\sin\theta$ into one cosine of a shifted angle.

Worked example – harmonic form (adapted from Edexcel IAL WMA13 June 2025 Q2). Express $f(x) = 7\cos x - 24\sin x$ in the form $R\cos(x+\alpha)$, $R > 0$, $0 < \alpha < 2\pi$.

- $R = \sqrt{7^2 + 24^2} = \sqrt{625} = 25$.
- Expand $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$ and match: $25\cos\alpha = 7$ and $25\sin\alpha = 24$, so $\tan\alpha = \frac{24}{7}$ and

$$\alpha = 1.287 \text{ (3 d.p.)} \qquad f(x) = 25\cos(x + 1.287)$$

- Check: $25\cos(0 + 1.287) = 25(0.28) = 7.0$ ✓ and $-25\sin(1.287) = -24.0$ ✓.

Worked example – identity proof (adapted from Edexcel IAL WMA13 June 2025 Q8a). Prove $\tan 3x \equiv \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

- Write $3x = 2x + x$ and use the **compound angle** formula:

$$\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}, \qquad \tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2t}{1 - t^2} \ (t = \tan x)$$

- Substitute:

$$\tan 3x = \frac{\frac{2t}{1-t^2} + t}{1 - \frac{2t^2}{1-t^2}} = \frac{2t + t(1-t^2)}{(1-t^2) - 2t^2} = \frac{3t - t^3}{1 - 3t^2} \qquad \blacksquare$$

- Check at $x = \frac{\pi}{6}$ ($t = \frac{1}{\sqrt{3}}$): numerator $\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}$; denominator $1 - 1 = 0$ – and $\tan\frac{\pi}{2}$ is undefined ✓ consistent.

Exponentials, logarithms and log-graph models

$y = e^x$ passes through $(0, 1)$ and grows without bound; $\ln x$ is its inverse with domain $x > 0$ and the y-axis as asymptote. Solve $e^{ax+b} = p$ by taking ln, and $\ln(ax + b) = q$ by exponentiating.

Log-graph models : for $y = ax^n$, taking logs gives $\log y = n \log x + \log a$ – plot $\log y$ against $\log x$ for a straight line with gradient n and intercept $\log a$. For $y = kb^x$, plot $\log y$ against x: gradient $\log b$, intercept $\log k$.

Worked example – recover a power law (adapted from Edexcel IAL WMA13 June 2025 Q3). A $\log_{10} y$–$\log_{10} x$ line passes through $(-3.5, 0)$ and $(0, -2)$. Express y as px^q.

- Gradient: $\dfrac{0-(-2)}{-3.5-0} = -\dfrac{4}{7}$; intercept -2:

$$\log_{10} y = -\tfrac{4}{7} \log_{10} x - 2$$

- Convert: $y = 10^{-\frac{4}{7}\log_{10} x - 2} = 10^{-2}\, x^{-4/7}$, so

$$y = \frac{1}{100}\, x^{-4/7} \qquad (p = \tfrac{1}{100},\ q = -\tfrac{4}{7})$$

- Check with a point on the line: $\log_{10} x = 0 \Rightarrow x = 1$, $\log_{10} y = -2 \Rightarrow y = \frac{1}{100}$ ✓ and the formula gives $y = \frac{1}{100} \cdot 1 = \frac{1}{100}$ ✓.

Worked example – exponential decay model (adapted from Edexcel IAL WMA13 June 2025 Q6). A processor cools as $\theta = 21 + Ae^{-kt}$ (θ in °C, t in minutes), starting at 75 °C, reaching 25 °C after 5 minutes; find A, k, and the time when the temperature falls at 9 °C per minute.

- At $t = 0$: $75 = 21 + A$, so $A = 54$.
- At $t = 5$: $25 = 21 + 54e^{-5k} \implies e^{-5k} = \frac{4}{54} = \frac{2}{27}$:

$$-5k = \ln \tfrac{2}{27} \implies k = \tfrac{1}{5} \ln \tfrac{27}{2} = 0.521 \text{ (3 s.f.)}$$

- Rate: $\dfrac{d\theta}{dt} = -54ke^{-kt}$; set equal to -9:

$$e^{-kt} = \frac{9}{54 \times 0.521} = \frac{9}{28.13} \implies T = \frac{\ln(28.13/9)}{0.521} = 2.19 \text{ (2 d.p.)}$$

- The **initial value** is 75 °C and the **long-run floor** is 21 °C.
- Check: at $T = 2.19$, $\theta = 21 + 54e^{-0.521(2.19)} = 21 + 54(0.320) = 38.3$ °C, falling at $28.13 \times 0.320 = 9.0$ °C/min ✓. The model's floor is 21 °C (room temperature) – sensible for large t.

Differentiation: rules, implicit and models

Differentiate $\sin kx$, $\cos kx$, $\tan kx$ ($\to k\sec^2 kx$), e^{kx}, $\ln kx$ ($\to \frac{1}{x}$), plus sums and differences.

Three rules extend the reach: the **product rule** , the **quotient rule** and the **chain rule** .

$$\text{product: } (uv)' = u'v + uv' \qquad \text{quotient: } \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \qquad \text{chain: } \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$\sec x \to \sec x\tan x$, $\operatorname{cosec} x \to -\operatorname{cosec} x\cot x$, $\cot x \to -\operatorname{cosec}^2 x$ follow from the quotient rule. **Implicit differentiation** uses $\dfrac{dy}{dx} = 1 \div \dfrac{dx}{dy}$ when x is given in terms of y.

Worked example – product and chain together (from the specification's own examples). Differentiate $e^{-x^2}\sin 3x$ (adapted from Edexcel IAL WMA13 June 2025 Q7).

- Product rule with $u = e^{-x^2}$ (chain: $u' = -2xe^{-x^2}$) and $v = \sin 3x$ ($v' = 3\cos 3x$):

$$\frac{dy}{dx} = -2xe^{-x^2}\sin 3x + 3e^{-x^2}\cos 3x = e^{-x^2}\big(3\cos 3x - 2x\sin 3x\big)$$

- Check at $x = 0$: derivative $= 1 \times (3 - 0) = 3$, and near 0 the curve behaves like $\sin 3x$ with gradient 3 ✓.

Integration: standard forms and recognition

Integrate the standard forms directly, and use **recognition forms** (spotting f' inside the integral):

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c \qquad \int f'(x)\big(f(x)\big)^n\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1} + c \quad (n \neq -1)$$

Trig identities convert $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ into integrable forms ($\cos 2A \equiv 1 - 2\sin^2 A$, $\cos 2A \equiv 2\cos^2 A - 1$; $\tan^2 \equiv \sec^2 - 1$).

Worked example – identity first (adapted from Edexcel IAL WMA13 June 2025 Q5). Find $\displaystyle\int \sin^2 3x\,dx$.

- **Double angle** with $A = 3x$: $\cos 6x \equiv 1 - 2\sin^2 3x$, so $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$:

$$\int \sin^2 3x\,dx = \tfrac{1}{2}\int (1 - \cos 6x)\,dx = \tfrac{1}{2}\left(x - \tfrac{1}{6}\sin 6x\right) + c = \frac{x}{2} - \frac{\sin 6x}{12} + c$$

- Check by differentiating: $\frac{1}{2} - \frac{6\cos 6x}{12} = \frac{1}{2}(1 - \cos 6x) = \sin^2 3x$ ✓.

Worked example – recognition. Find $\displaystyle\int \frac{6x}{3x^2+4}\,dx$.

- The numerator is the derivative of the denominator ($f' = 6x$, $f = 3x^2 + 4$):

$$\int \frac{6x}{3x^2+4}\,dx = \ln|3x^2+4| + c$$

- Check: $\dfrac{d}{dx}\ln(3x^2+4) = \dfrac{6x}{3x^2+4}$ ✓. The powered form works the same way: with $u = 3x^2 + 4$ and $du = 6x\,dx$,

$$\int x\,(3x^2+4)^4\,dx = \tfrac{1}{6}\int u^4\,du = \frac{(3x^2+4)^5}{30} + c$$

  and differentiating $\dfrac{(3x^2+4)^5}{30}$ gives $\dfrac{5(3x^2+4)^4 \cdot 6x}{30} = x(3x^2+4)^4$ ✓.

Numerical methods: sign changes and iteration

- **Sign change** : if f is continuous on $[a, b]$ and $f(a)f(b) < 0$, a root lies between a and b. Halve the interval to pin a root to any accuracy.
- **Iteration** : a rearrangement $x_{n+1} = f(x_n)$ from a start value converges to a **fixed point** (or not – always check against a sign-change bound).

Worked example – locate and iterate (adapted from Edexcel IAL WMA13 June 2025 Q7). The stationary point of $y = e^{-x^2}\sin 3x$ near $x = 0.4$ satisfies $x = \frac{1}{3}\arctan\frac{3}{2x}$.
(a) With $x_1 = 0.4$:

$$x_2 = \tfrac{1}{3}\arctan\left(\tfrac{3}{0.8}\right) = \tfrac{1}{3}\arctan(3.75) = 0.4367 \qquad x_4 = 0.4307 \text{ (4 d.p.)}$$

- Check the iteration converges: $|x_3 - x_2|$ small and the values settle toward 0.43 ✓.
  (b) Show the root is 0.430 to 3 decimal places: define $g(x) = x - \frac{1}{3}\arctan\frac{3}{2x}$ (or the derivative expression from the worked example above):

$$g(0.4295) = -1.1 \times 10^{-3} < 0 \qquad g(0.4305) = +6.4 \times 10^{-5} > 0$$

- g is continuous, and the sign change between 0.4295 and 0.4305 brackets a root, so the root is 0.430 to 3 d.p. ✓.

Checklist before you call this unit done

- Simplify rational expressions; factorise before cancelling; divide by linear or quadratic divisors.
- Work with domain, range, composition and inverses; verify inverses by composing to x.

- Sketch and solve modulus equations by cases with checks; combine two transformations on one graph.
- Use sec, cosec, cot and their identities; know the inverse-function restricted domains.
- Convert $a\cos\theta + b\sin\theta$ to $R\cos(\theta \pm \alpha)$; read off maxima and solve.
- Prove compound and double angle identities; use them forwards and backwards.
- Solve $e^{ax+b} = p$ and $\ln(ax+b) = q$; recover $y = ax^n$ and $y = kb^x$ from log plots.
- Differentiate with product, quotient and chain rules, including sec / cosec / cot; use $\frac{dy}{dx} = 1 \div \frac{dx}{dy}$.
- Model growth and decay; interpret the initial value and long-run floor; critique the model.
- Integrate standard forms; spot $\int f'/f$ and $\int f'(f)^n$; use identities on squared trig functions.
- Locate roots by sign change with a continuity statement; run and check an iteration.