Solutions

2.1 $2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$.

2.2 $2x + 2y + 2x\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x+y}{x}$.

2.3 $V = x^3$: $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt} = 3(25)(0.1) = 7.5$ cm^3/s.

2.4 $\int y\,dy = \int 2x\,dx \Rightarrow \frac{y^2}{2} = x^2 + c$; $y(0) = 3$ gives $c = \frac{9}{2}$, so $y^2 = 2x^2 + 9$.

3.1 $V = \frac{4}{3}\pi r^3$, $\dfrac{dV}{dr} = 4\pi r^2$; chain rule with $\dfrac{dV}{dt} = -12$:

$$-12 = 4\pi r^2\frac{dr}{dt} \implies \frac{dr}{dt} = -\frac{3}{\pi r^2}$$

The rate of decrease $\dfrac{3}{\pi r^2}$ is a constant multiple of $\dfrac{1}{r^2}$ – inversely proportional to r^2. ■
At $r = 3$:

$$\frac{3}{9\pi} = \frac{1}{3\pi} \approx 0.106 \text{ cm/s (3 s.f.)}$$

Check: units cm^3/s ÷ cm^2 = cm/s ✓.

3.2 Separate: $\displaystyle\int y\,dy = \int 6\sec 3x\tan 3x\,dx$ with $\int \sec 3x\tan 3x\,dx = \frac{1}{3}\sec 3x$:

$$\frac{y^2}{2} = 2\sec 3x + c \implies y^2 = 4\sec 3x + C$$

Condition $y = 2$ at $x = 0$: $4 = 4(1) + C \Rightarrow C = 0$:

$$y^2 = 4\sec 3x$$

Check: $\sqrt{4} = 2$ at $x = 0$ ✓; differentiating back: $2y\dfrac{dy}{dx} = 12\sec 3x\tan 3x \Rightarrow y\dfrac{dy}{dx} = 6\sec 3x\tan 3x$ ✓.

Local review guidance: correct implicit/rate structure; exact derivative or particular solution; condition applied and the solution checked. This is authored classroom guidance.

3.3

(a)

- similar triangles: $\dfrac{r}{h} = \dfrac{3}{6}$, so the water surface has radius $r = \frac{1}{2}h$
- volume of a cone, $V = \frac{1}{3}\pi r^2 h$:

$$V = \tfrac{1}{3}\pi\left(\tfrac{1}{2}h\right)^2 h = \frac{\pi h^3}{12}$$

(b)

- chain rule, $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt}$, with $\dfrac{dV}{dh} = \dfrac{\pi h^2}{4}$:

$$\frac{\pi h^2}{4}\frac{dh}{dt} = -kh^{\frac{1}{2}} \Rightarrow h^{\frac{3}{2}}\frac{dh}{dt} = -\frac{4k}{\pi}$$

(c)

- separate the variables and integrate:

$$\int h^{\frac{3}{2}}\,dh = \int -\frac{4k}{\pi}\,dt \Rightarrow \tfrac{2}{5}h^{\frac{5}{2}} = -\frac{4k}{\pi}t + c$$

- $t = 0$, $h = 4$: $c = \frac{2}{5}(32) = 12.8$; $t = 31$, $h = 1$: $\frac{2}{5} = -\dfrac{4k}{\pi}(31) + 12.8$, so $\dfrac{4k}{\pi} = 0.4$ and $k = \mathbf{0.1}\pi$
- empty when $h = 0$: $0 = -0.4t + 12.8$, so $t = \mathbf{32}$ minutes