

4.3 The binomial series for rational powers

Name: _____ Class: _____ Date: _____ Total: 25 marks

KEY WORDS binomial series • validity • rational

Objective

Expand $(a + bx)^n$ for any rational n with the validity condition, combine expansions via partial fractions, and use expansions for approximations. Original teaching questions for specification item P4.4.1; the exam-style part adapts reviewed WMA14 June 2025 demand.

1 Key knowledge and worked examples

The **binomial series** for any rational n , valid for $|x| < 1$:

$$(1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!}x^2 + \frac{n(n - 1)(n - 2)}{3!}x^3 + \dots$$

For $(a + bx)^n = a^n \left(1 + \frac{b}{a}x\right)^n$, valid $|x| < \left|\frac{a}{b}\right|$. The series continues indefinitely unless n is a positive integer; reject x values outside the validity. Partial fractions first when the expression is rational.

Worked example (adapted from Edexcel IAL WMA14 June 2025 Q8). Expand $\sqrt{4 + x}$ in ascending powers of x to three non-zero terms.

- Factor: $2 \left(1 + \frac{x}{4}\right)^{1/2}$, valid $|x| < 4$:

$$\sqrt{4 + x} = 2 \left[1 + \frac{1}{2} \frac{x}{4} + \frac{\frac{1}{2}(-\frac{1}{2})}{2} \left(\frac{x}{4}\right)^2 + \dots \right] = 2 + \frac{x}{4} - \frac{x^2}{64} + \dots$$

- Check at $x = 1$: $2 + 0.25 - 0.0156 = 2.2344$ against $\sqrt{5} = 2.2361$ ✓ (2 d.p.).

Worked example – negative n . Expand $(1 - 2x)^{-2}$ to three terms, $|x| < \frac{1}{2}$.

$$1 + (-2)(-2x) + \frac{(-2)(-3)}{2}(-2x)^2 = 1 + 4x + 12x^2 + \dots$$

- Check at $x = 0.1$: series $1 + 0.4 + 0.12 = 1.52$; true $(0.8)^{-2} = 1.5625$ ✓ (converging).

2 Practice

2.1 Expand $(1 + x)^{\frac{1}{3}}$ to three terms. [3]

2.2 Expand $(2 - x)^{-1}$ to three terms and state the validity. [3]

2.3 Expand $\sqrt{9 + x}$ to three terms. [3]

2.4 Use your answer to 2.3 to estimate $\sqrt{9.4}$ to 4 decimal places. [2]

3 Exam-style questions

3.1 (Adapted from Edexcel IAL WMA14 June 2025 Q8.)

- (a) Find the first three non-zero terms of the binomial series expansion of $\sqrt{4+x}$ in ascending powers of x , giving each coefficient as a simplified fraction.
- (b) Write down the first three non-zero terms of the expansion of $\sqrt{4-x}$.
- (c) Hence show that $\sqrt{\frac{4+x}{4-x}} \approx 1 + \frac{x}{4}$, and use it with $x = 1$ to give a rational approximation to one decimal place check of $\sqrt{\frac{5}{3}}$.

3.2 Express $\frac{7-x}{(2-x)(1+x)}$ in partial fractions and hence expand it in ascending powers of x to three terms, stating the validity.
