

Solutions

2.1 $y' = 2x - 8 = 0 \implies x = 4$; $y(4) = -13$; $y'' = 2 > 0$: minimum at $(4, -13)$.

2.2 $y' = 4x^3 - 4x = 4x(x-1)(x+1) = 0$: $x = 0, \pm 1$. $y(\pm 1) = 1 - 2 = -1$; $y(0) = 0$.
 $y'' = 12x^2 - 4$: at ± 1 , $8 > 0 \rightarrow$ minima $(\pm 1, -1)$; at 0 , $-4 < 0 \rightarrow$ maximum $(0, 0)$.

2.3 $y' = 3x^2 - 6x = 3x(x-2)$: increasing for $x < 0$ or $x > 2$; decreasing for $0 < x < 2$.

2.4 $y' = 1 - \frac{1}{x^2} = 0 \implies x = 1$ (given $x > 0$). $y(1) = 2$; $y'' = \frac{2}{x^3} > 0$: minimum at $(1, 2)$.

2.5 $y' = 6x^2 + 18x - 24 = 6(x+4)(x-1) = 0$: $x = 1$ and $x = -4$. $y(1) = 2 + 9 - 24 = -13$; $y(-4) = -128 + 144 + 96 = 112$. Points $(1, -13)$ and $(-4, 112)$.

3.1 (a) Volume: $x^2L = 20$, so $L = \frac{20}{x^2}$ and

$$S = 2x^2 + 4x \cdot \frac{20}{x^2} = 2x^2 + \frac{80}{x}$$

(b) $\frac{dS}{dx} = 4x - \frac{80}{x^2} = 0 \implies 4x^3 = 80 \implies x = \sqrt[3]{20} = 2.71$ (3 s.f.).

$$S = 2(20)^{2/3} + \frac{80}{(20)^{1/3}} = 14.74 + 29.48 = 44.2 \text{ m}^2 \text{ (1 d.p.)}$$

(c) $\frac{d^2S}{dx^2} = \frac{160}{x^3} > 0$ for all $x > 0$ (at $x = \sqrt[3]{20}$ it equals $\frac{160}{20} = 8 > 0$), so the stationary point is a minimum ✓.

3.2 $y' = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$: $x = 1, 3$. $y(1) = 1 - 6 + 9 = 4$; $y(3) = 27 - 54 + 27 = 0$. $y'' = 6x - 12$: at 1 , $-6 < 0 \rightarrow$ maximum $(1, 4)$; at 3 , $6 > 0 \rightarrow$ minimum $(3, 0)$. Sketch: crosses the x -axis at $x = 0$ and touches at $x = 3$ (minimum on the axis), rises to $(1, 4)$, falls to $(3, 0)$, then increases – a positive cubic through the origin. Check: $y(0) = 0$ ✓.

Local review guidance: derivative set to zero after eliminating the second variable; exact coordinates; nature justified by second derivative or sign change. This is authored classroom guidance.